

# Kodaira Families and Newton-Cartan Structures with Torsion

James Gundry

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences,  
Wilberforce Road, Cambridge, CB3 0WA

## Abstract

We describe the induced geometry on several classes of Kodaira moduli spaces of rational curves in twistor spaces. By constructing connections and frames on the moduli spaces we build and review twistor theories pertaining to relativistic and non-relativistic geometries. Focussing on the cases of three- and five-dimensional moduli spaces we establish novel twistor theories of Newton-Cartan spacetimes. We generalise a canonical class of connections on Kodaira moduli spaces to encompass torsion and prove that in three dimensions deformations of the twistor space's holomorphic structure induce Newton-Cartan structures with torsion. In five dimensions the canonical connections contain generalised Coriolis forces, and here also we consider the introduction of torsion by deformation. Non-relativistic limits are exhibited as jumping phenomena of normal bundles. We consider jumping phenomena in Newtonian twistor theory, as well as exhibiting via twistor theory that three-dimensional torsional Newton-Cartan geometry generically exists on the jumping hypersurfaces of Gibbons-Hawking manifolds.

J.M.GUNDRY@DAMTP.CAM.AC.UK

DAMTP-2017-15

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Induced Metrics . . . . .	2
1.2	Induced Affine Connections . . . . .	4
1.2.1	The $\Xi$ -Connection . . . . .	5
1.2.2	The $\Lambda$ -Connection . . . . .	6
1.2.3	The Torsion $\Xi$ -Connection . . . . .	7
1.3	Newton-Cartan Geometry . . . . .	8
<b>2</b>	<b>Examples of Frames and Connections</b>	<b>10</b>
2.1	Line Bundles on $\mathbb{P}^1$ . . . . .	10
2.1.1	Odd Dimensions . . . . .	10
2.1.2	Even Dimensions . . . . .	11
2.2	$\mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1) \rightarrow \mathbb{P}^1$ . . . . .	12
2.3	Limits in $4n$ Dimensions . . . . .	13
<b>3</b>	<b>Newtonian Twistor Theory in Three Dimensions</b>	<b>15</b>
3.1	Galilean Structures and Canonical Connections . . . . .	15
3.2	Deformations and Torsion . . . . .	19
3.3	Global Vectors . . . . .	25
3.4	The Relativistic Limit . . . . .	26
3.5	On Jumping Hypersurfaces of Gibbons-Hawking Manifolds . . . . .	27
<b>4</b>	<b>Some Novel Features in Four Dimensions</b>	<b>30</b>
4.1	The $\Xi$ -Connection for $Z = \mathcal{O} \oplus \mathcal{O}(2)$ . . . . .	30
4.2	Jumps in Four-Dimensional Newtonian Twistor Theory . . . . .	30
<b>5</b>	<b>Five Dimensions</b>	<b>33</b>
5.1	Galilean Structures and Canonical Connections . . . . .	33
5.2	Deformations in the Newtonian Theory . . . . .	37
5.3	Global Vectors . . . . .	40
5.4	Alpha-Surfaces in the Relativistic Theory . . . . .	40

# 1 Introduction

Let  $Z$  be a complex manifold. One can study the moduli space  $M$  of a family of compact complex submanifolds  $X_x \subset Z$  for  $x \in M$ , and remarkably one finds that  $M$  often comes equipped with canonical geometrical structures such as metrics, forms, and affine connections.

**Definition 1.1.** [1] *A holomorphic family of compact complex submanifolds embedded in a complex manifold  $Z$  with moduli space  $M$  is a submanifold  $F \subset Z \times M$  such that the restriction  $\nu : F \rightarrow M$  of the natural projection  $Z \times M \rightarrow M$  is a proper regular map, and the submanifolds in  $Z$  are  $X_x = \mu(\nu^{-1}(x))$  for  $x \in M$ , where  $\mu : F \rightarrow Z$  is the restriction to  $F$  of the natural projection  $Z \times M \rightarrow Z$ .*

The submanifold  $F$  is referred to as the correspondence space for the construction. Kodaira showed when such a situation can occur, giving a cohomological test for the submanifolds.

**Theorem 1.2.** [5, 1] *Let  $X_0 \subset Z$  be a compact complex embedded submanifold of a complex manifold  $Z$  such that  $\check{H}^1(X_0, N_x) = 0$  (where  $N_x = T^*Z|_{X_0}/TX_0$  is its normal bundle). Then  $X_0$  is a member of a maximal holomorphic family  $F$  with moduli space  $M$ , and this family is complete in the sense that there is an isomorphism  $T_x M = \check{H}^0(X_0, N_x)$ .*

The dimension of  $M$  is thus equal to the dimension of  $\check{H}^0(X_0, N_x)$ . When the submanifolds are rational curves  $X_x = \mathbb{P}^1$  we'll call the construction a twistor theory, and  $Z$  is a twistor space. The correspondence space is then  $F = M \times \mathbb{P}^1$ , which can be identified with the projective primed spin bundle  $PS'$ , the quotient by a homogeneity operator  $\Upsilon = \pi_{A'} \frac{\partial}{\partial \pi_{A'}}$  of a rank-two holomorphic vector bundle  $S'$  with coordinates  $\pi_{A'}$  on fibres. The most famous example of this construction is the nonlinear graviton construction of Penrose [6], in which  $Z$  is a complex three-fold containing a rational curve  $X_0$  with normal bundle  $N_0 = \mathcal{O}(1) \oplus \mathcal{O}(1)$ ;  $M$  is then a four-dimensional anti-self-dual conformal manifold, and can be promoted to an Einstein manifold by imposing extra requirements. The moduli space is naturally complex, but real slices with Euclidean or neutral signature can be taken [20]. Other examples are the cases

- $\dim Z = 2$  with  $N_0 = \mathcal{O}(2)$ , where  $M$  is a three-dimensional Einstein-Weyl manifold [3];
- $\dim Z = 2$  with  $N_0 = \mathcal{O}(1)$ , where  $M$  is a two-dimensional projective manifold [3];
- and  $\dim Z = 3$  with  $N_0 = \mathcal{O} \oplus \mathcal{O}(2)$ , where  $M$  is a four-dimensional Newton-Cartan manifold [2].

In this paper we will add some new members to the list above by exhibiting some novel Kodaira families with  $X_x = \mathbb{P}^1$ . Highlights will include new Newtonian twistor spaces, and in section 3.1 we'll prove the following.

**Theorem 3.1.** *Let  $Z = \mathcal{O} \oplus \mathcal{O}(1)$  with global sections  $X_x$ . The moduli space  $M \ni x$  of these rational curves is a complex three-dimensional manifold equipped with a family of Newton-Cartan structures parametrised by three arbitrary functions on  $M$  and an element of  $GL(2, \mathbb{C})$ .*

We thus establish a flat model for the twistor theory of three-dimensional Newton-Cartan spacetimes. A certain class of deformations of the complex structure turn out to be interesting, inducing torsion. In section 3.2 we will prove the following.

**Theorem 3.3.** *Let  $Z$  be the total space of an affine line bundle on  $\mathcal{O}(1)$  with trivial underlying translation bundle whose patching is*

$$\hat{T} = T + f$$

*where  $f$  represents a cohomology class in  $\check{H}^1(\mathcal{O}(1), \mathcal{O}_{\mathcal{O}(1)})$ . The three-parameter family of global sections  $X_x$  have normal bundle  $X_x = \mathcal{O} \oplus \mathcal{O}(1)$  and the moduli space  $M$  of those sections is a complex three-dimensional manifold equipped with a family of torsional Newton-Cartan structures parametrised by two arbitrary one-forms on  $M$ , two functions on  $M$ , and an element of  $GL(2, \mathbb{C})$ .*

To prove this one must generalise the construction of affine connections on Kodaira moduli spaces to include torsion; we make this generalisation in section 1.2.3. Torsion in Newton-Cartan geometry is a subject which has attracted significant recent interest [18, 27, 28], and is characterised by the non-closure of the Newton-Cartan clock (see section 1.3 for an introduction to the relevant definitions). Analogous theorems for the case of five-dimensional Newton-Cartan spacetimes will also be proven, for which the twistor spaces are four-dimensional and the isomorphism class of the normal bundles is  $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ .

We'll also discuss some Kodaira families whose geometry is not Newton-Cartan, considering the induced geometry on families of rational curves with normal bundles  $\mathcal{O}(2n)$ ,  $\mathcal{O}(2n - 1)$ , and  $\mathcal{O}(2n - 1) \oplus \mathcal{O}(2n - 1)$  for  $n \geq 1$ . The emphasis will be on the construction of the frame on  $M$ , as well as two canonical families of affine connections. Some of these examples have been considered previously in the literature, as will be described in section 2, and are included as examples of the frame and connection constructions advocated in this paper. We will also discuss some of the links between the above examples in terms of jumping phenomena.

The layout of the paper is as follows. In the remainder of this section we will describe the methods of constructing induced geometry on Kodaira families which will be preferred in this paper: we will consider the construction of frame one-forms, and following Merkulov [1], the construction of two kinds of canonical affine connections. We then proceed to generalise one of these, the  $\Xi$ -connection, to include torsion. In section 2 we'll apply these ideas to some simple Kodaira families, finding "relativistic" induced geometry in various numbers of dimensions and discussing some Newtonian limits. Section 3 will describe the development of a twistor theory of three-dimensional Newton-Cartan manifolds, including the discussion of the introduction of torsion by Kodaira deformations. Section 4 will introduce a couple of supplementary results on Newtonian twistor theory in four dimensions. Finally in section 5 the material will be extended to five dimensions, where we will find that the induced Newton-Cartan geometry comes naturally equipped with (self-dual) Coriolis forces.

## 1.1 Induced Metrics

Given a Kodaira family of rational curves there are various ways of constructing the induced geometry on  $M$ , all relying on the isomorphism from theorem 1.2. In Penrose's original approach one looks at the explicit alpha surfaces induced on  $M$  by the pull-back of the twistor coordinates to the correspondence space and then uses the alpha surfaces to single out preferred vectors [6]. Alternatively, in the case of the nonlinear graviton, one can directly find the contravariant conformal structure as (the image on  $M$  of) the zero section of  $\check{H}^0(Z, TZ \odot TZ)$ . Yet a third approach would be to construct an integral formula from twistor cohomology classes and then compute the differential operator(s) on  $M$  for which the twistor classes constitute the kernel.

In this paper we'll employ yet another method, in which (families of) frames of one-forms are directly computed on  $M$ .

**Theorem 1.3.** *Let  $M$  be the moduli space of a complete holomorphic family of rational curves  $X_x = \mathbb{P}^1$  in a complex manifold  $Z \rightarrow \mathbb{P}^1$  with normal bundles  $N_x \rightarrow \mathbb{P}^1$ .  $M$  is equipped with a preferred family of one-forms - the frame - defined uniquely up to an element of  $\check{H}^0(X_x, N_x \otimes N_x^*)$  per section  $X_x$ .*

### Proof

Let  $Z$  be  $(k+1)$ -dimensional. The family of one-forms on the moduli space  $M$  arises as a section of  $N_x \otimes \Lambda_x^1(M)$  for each  $x \in M$ . Cover  $\mathbb{P}^1$  by two patches  $U$  and  $\hat{U}$  with coordinates  $\lambda$  and  $\hat{\lambda}$  respectively, with holomorphic transition function

$$\hat{\lambda} = \lambda^{-1}$$

on  $U \cap \hat{U}$ . We can describe  $Z \xrightarrow{\pi} \mathbb{P}^1$  concretely as a complex manifold by exhibiting its patching. If  $\hat{w}^\mu(w^\nu, \lambda)$  is this patching (for  $\mu, \nu = 0, 1, \dots, k$ ) then we can describe the global section  $X_x$  (over  $U$ ) by the equation  $w^\mu = w^\mu| (x, \lambda)$  for  $x \in M$  parametrising the space of sections and where  $w^\mu|$  are functions extracted from the patching.

One then finds that

$$d\hat{w}^\mu| = \mathcal{F}_\nu^\mu dw^\nu|$$

where

$$\mathcal{F}_\nu^\mu(x, \lambda) = \frac{\partial \hat{w}^\mu}{\partial w^\nu}(w^\alpha|, \lambda)$$

is the patching of the normal bundle  $N_x$ . (For notational convenience we will use a vertical slash to denote the restriction to  $X_x$  throughout this paper.) By the Birkhoff-Grothendieck theorem [16] we can write

$$\mathcal{F} = \hat{H} \text{diag}(\lambda^{-n_1}, \lambda^{-n_2}, \dots, \lambda^{-n_k}) H^{-1}$$

where  $H$  and  $\hat{H}$  are holomorphic maps to  $\text{GL}(k, \mathbb{C})$  from  $U$  and  $\hat{U}$  respectively, and where the integers  $(n_1, n_2, \dots, n_k)$  specify the isomorphism class of  $N$ . We then extract the frame from a section  $(H^{-1})_\nu^\mu dw^\nu|$  (or equivalently  $(\hat{H}^{-1})_\nu^\mu d\hat{w}^\nu|$ ), and the redundancy consists in multiplying  $H$  (and  $\hat{H}$ ) by a global section of  $N_x \otimes N_x^*$  (which may vary arbitrarily with  $x \in M$ ). We denote

$$v(x) = H^{-1}dw| \in \check{H}^0(F|_x, N_x \otimes \Lambda_x^1(M))$$

the *frame section*.

The frame section then gives rise to a collection of one-forms  $e^{AB\dots CA'B'\dots C'}$  on  $M$ , where the unprimed indices arise from the component-structure of  $v$  and  $A'B'C'$  (each running from  $0'$  to  $1'$ ) arise from the dependence of  $v$  on the base  $\mathbb{P}^1$ . For example, in the case where

$$N_x = \mathcal{O}(2) \oplus \mathcal{O}(2)$$

we write

$$v^A = e^{AA'_1A'_2} \pi_{A'_1} \pi_{A'_2},$$

where  $[\pi_{A'}]$  are homogeneous coordinates on the base  $\mathbb{P}^1$  and where  $A = 0, 1$ . One then extracts the frame  $e^{AA'_1A'_2} = e_a^{AA'_1A'_2}(x^b) dx^a$ .  $\square$

In this paper we will include in  $v$  the result of the most general element of  $\check{H}^0(X_x, N_x \otimes N_x^*)$  per line  $X_x$  in the guise of arbitrary functions. (This is analogous to writing a conformal structure  $[g]$  as a single metric  $g = \alpha g_0$  for some representative  $g_0 \in [g]$  and some arbitrary non-vanishing function  $\alpha$ . In some cases this analogy is an identity.)

**Definition 1.4.** *All tensor fields on  $M$  which can be constructed from the frame using only the tensor product and the symplectic forms  $\epsilon_{AB}$  and  $\epsilon_{A'B'}$  are induced on  $M$  as the span of  $v$ .*

Often, pleasingly, the span of  $v$  contains a metric (which may depend on some number of arbitrary functions), and in the nonlinear graviton construction this metric is exactly the conformal structure we could induce in a more traditional way (i.e. via the presence of alpha surfaces in  $M$  as is done in [6]).

Using alpha surfaces to find the induced conformal structure (or otherwise) becomes increasingly complicated in higher dimensions, but the frame method of theorem 1.3 does not. We will thus make frequent use of it, though without losing sight completely of the existence of alpha surfaces.

## 1.2 Induced Affine Connections

The construction of affine connections on the moduli spaces of complete holomorphic families of submanifolds was considered by Merkulov [7, 1]. This work led to the solution of the holonomy problem [8].

First consider torsion-free affine connections more generally. Let  $J_x^k$  be the ideal of germs of holomorphic functions on  $M$  which vanish to order  $k$  at  $x \in M$ . The second-order tangent bundle  $T^{[2]}M$  is defined to be the union over all points in  $M$  of second-order tangent spaces

$$T_x^{[2]}M = (J_x/J_x^3)^*.$$

An element  $(V^{ab}, V^a)$  of  $T_x^{[2]}M$  consists of the first two non-vanishing terms of the Taylor expansion of a function vanishing at  $x$ ; a section of  $T^{[2]}M$  gives rise to a second-order linear differential operator

$$V^{[2]} = (V^{ab}, V^a) \rightsquigarrow V^{ab}\partial_a\partial_b + V^a\partial_a,$$

where for brevity's sake we put  $\partial_a = \frac{\partial}{\partial x^a}$ . There is a short exact sequence

$$0 \rightarrow TM \rightarrow T^{[2]}M \rightarrow \odot^2 TM \rightarrow 0 \quad (1)$$

with maps

$$(V^a) \mapsto (0, V^a) \quad \text{and} \quad (V^{ab}, V^a) \mapsto (V^{ab}).$$

A torsion-free affine connection  $\nabla$  on  $TM$  is then equivalent to a (left) splitting of (1), i.e. a linear map

$$\gamma : T^{[2]}M \rightarrow TM \quad (2)$$

acting as

$$\gamma : (V^{ab}, V^a) \mapsto (V^a + \Gamma_{bc}^a V^{bc})$$

for functions  $\Gamma_{bc}^a$  on  $M$  which we identify as Christoffel symbols. We'll now describe, following [1], two ways of constructing such a map in twistor theory. The original treatment in [1] is considerably more sophisticated than what is given here, as Merkulov describes the construction for a general Kodaira moduli space. In the case  $Z \rightarrow \mathbb{P}^1$  the construction is much simpler.

### 1.2.1 The $\Xi$ -Connection

As above cover  $\mathbb{P}^1$  with two open sets  $U$  and  $\hat{U}$  with respective coordinate functions  $\lambda$  and  $\hat{\lambda}$  subject to the transition function  $\hat{\lambda} = \lambda^{-1}$  on  $U \cap \hat{U}$ . These coordinates are known as *inhomogeneous* coordinates on  $\mathbb{P}^1$ . Again consider a general fibred twistor space  $\pi : Z \rightarrow \mathbb{P}^1$  characterised by the holomorphic patching

$$\hat{w}^\mu = \hat{w}^\mu(w^\nu, \lambda), \quad (3)$$

where  $w^\mu$  are the twistor coordinates on the fibres. Henceforth restrict to cases in which one has

$$\check{H}^1(\mathbb{P}^1, N_x \otimes N_x^*) = 0.$$

A section  $V = (V^{ab}, V^a)$  of  $T^{[2]}M$  gives rise to a differential operator  $V^{ab}\partial_a\partial_b + V^a\partial_a$  which following [1] we'll now apply to (3).

$$V^{ab}\partial_a\partial_b\hat{w}^\mu + V^a\partial_a\hat{w}^\mu = V^{ab}\mathcal{F}_\nu^\mu\partial_a\partial_b w^\nu + V^a\mathcal{F}_\nu^\mu\partial_a w^\nu + V^{ab}\mathcal{F}_{\nu\rho}^\mu\partial_a w^\nu\partial_b w^\rho, \quad (4)$$

where again we have

$$\mathcal{F}_\nu^\mu = \frac{\partial\hat{w}^\mu}{\partial w^\nu} \Big|$$

and

$$\mathcal{F}_{\nu\rho}^\mu = \frac{\partial^2\hat{w}^\mu}{\partial w^\nu\partial w^\rho} \Big|.$$

(Recall that a vertical slash indicates that one must restrict to global sections  $w^\nu = w^\nu|_{(x^a, \lambda)}$ .) If we can write this line as a global section of  $N_x$  then we have (via the Kodaira isomorphism  $T_x M = \check{H}^0(\mathbb{P}^1, N_x)$ ) constructed a map  $\gamma$ . The only problematic term is the last one. The  $\Xi$ -connection is constructed by splitting

$$\mathcal{F}_{\nu\rho}^\mu\partial_a w^\nu = -\hat{\chi}_{\alpha a}^\mu \mathcal{F}_\rho^\alpha + \mathcal{F}_\alpha^\mu \chi_{\rho a}^\alpha \quad (5)$$

for a 0-cochain  $\{\chi\}$  of  $N_x \otimes N_x^* \otimes T_x^* M$  per point  $x \in M$ . The left-hand-side of (5) is always (in the  $Z \rightarrow \mathbb{P}^1$  case) a 1-cocycle of  $N_x \otimes N_x^* \otimes T_x^* M$ , and since by assumption  $\check{H}^1(\mathbb{P}^1, N_x \otimes N_x^*) = 0$  the splitting is always possible.

Equation (4) then becomes

$$V^{ab}\partial_a\partial_b\hat{w}^\mu + V^a\partial_a\hat{w}^\mu + V^{ab}\hat{\chi}_{\alpha(a)}^\mu\partial_b\hat{w}^\alpha = \mathcal{F}_\nu^\mu (V^{ab}\partial_a\partial_b w^\nu + V^a\partial_a w^\nu + V^{ab}\chi_{\rho(a)}^\nu\partial_b w^\rho)$$

and so we have constructed a global section of  $N_x$  per point  $x$ . The connection symbols for the map thus constructed can be read off as

$$\partial_a\partial_b w^\nu + \chi_{\rho(a)}^\nu\partial_b w^\rho = \Gamma_{ab}^c\partial_c w^\nu.$$

There is, though, a possible source of non-uniqueness. If  $\check{H}^0(\mathbb{P}^1, N_x \otimes N_x^*) \neq 0$  then one is free to add any element of that group to both  $\hat{\chi}_{\alpha a}^\mu$  and  $\chi_{\alpha a}^\mu$ . Therefore what one obtains is an equivalence class of connections (the  $\Xi$ -connection). As with the frame we will choose to describe the equivalence class by a single most-general representative containing arbitrary functions.

## Example

Consider  $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$  with patching

$$\hat{\Omega}^A = \lambda^{-1} \Omega^A \quad \hat{\lambda} = \lambda$$

for  $A = 0, 1$  and global sections

$$\Omega^A| = x^{A0'} \lambda + x^{A1'}$$

where  $x^{AA'}$  are coordinates on  $M$  parametrising the family of global sections. (This is the twistor space for flat four-dimensional spacetime.)

The Christoffel symbols for the  $\Xi$ -connection are then given by

$$\Gamma^{AA'}_{CC'DD'} = \chi^A_{DCC'} \delta^{A'}_{D'} + \chi^A_{CDD'} \delta^{A'}_{C'} \quad , \quad (6)$$

where  $\chi^A_B$  are four arbitrary one-forms on  $M$  constituting a global section of  $N_x \otimes N_x^* \otimes T_x^* M$  per point  $x \in M$ .

### 1.2.2 The $\Lambda$ -Connection

An alternative twistor construction of a class of maps (2) gives rise to the so-called  $\Lambda$ -connection, which we now briefly summarise. This class of connections often degenerates into a single affine connection, and in as much as it is necessary and useful to make such a distinction, it is the  $\Lambda$ -connection which should be considered the *physical* connection.

Consider again the patching for a general fibred twistor space  $Z \rightarrow \mathbb{P}^1$ :

$$\hat{w}^\mu = \hat{w}^\mu(w^\nu, \lambda) \quad ,$$

where  $w^\mu$  are the coordinates on the fibres, and again consider the equation (4) resulting from the action of the section  $V$  of  $T^{[2]}M$ . To construct the  $\Lambda$ -connection we do the splitting differently.

We instead choose to solve

$$\mathcal{F}^\mu_{\alpha\beta} = -\hat{\sigma}^\mu_{\nu\rho} \mathcal{F}^\nu_\alpha \mathcal{F}^\rho_\beta + \mathcal{F}^\mu_\eta \sigma^\eta_{\alpha\beta} \quad (7)$$

for a 0-cochain  $\{\sigma\}$  of  $N \otimes (N^* \odot N^*) \rightarrow \mathbb{P}^1$ , and the Christoffel symbols for the resulting map  $\gamma$  can be read off from

$$\Gamma^a_{bc} \partial_a w^\mu| = \partial_b \partial_c w^\mu| + \sigma^\mu_{\nu\rho} \partial_b w^\nu| \partial_c w^\rho| \quad (8)$$

(or the equivalent expression over  $\hat{U}$ ). In the  $Z \rightarrow \mathbb{P}^1$  case the left hand side of (7) is always a 1-cocycle of  $N \otimes (N^* \odot N^*)$ .

The difficult part of this construction is the solution of the splitting problem (7), which in some cases is not possible, and is often not unique.

Uniqueness is determined by whether there are global sections of  $N \otimes (N^* \odot N^*)$ ; if these exist then one is free to add one to  $\{\sigma\}$  and so construct a different connection. In Penrose's case we have

$$\check{H}^0(\mathbb{P}^1, N \otimes (N^* \odot N^*)) = 0$$

and so the connection is always unique. This is unsurprising; we can always call upon the Levi-Civita connection.



There are Kodaira deformations (giving rise to  $\mathcal{F}_{\alpha\beta}^\mu$ ) for which (7) cannot be solved iff

$$\check{H}^1(\mathbb{P}^1, N \otimes (N^* \odot N^*)) \neq 0,$$

and there are several reasons this may occur. One is that the spacetime suffers a jump in the normal bundle; another is when the torsion-free requirement essential to the construction is broken. In Penrose's case we can calculate that  $\check{H}^1(\mathbb{P}^1, N \otimes (N^* \odot N^*))$  vanishes, so all Kodaira deformations lead to torsion-free connections, in line with the nonlinear graviton construction.

### 1.2.3 The Torsion $\Xi$ -Connection

In sections 3.2 and 5.2 the  $\Lambda$ -connections fail to exist and the  $\Xi$ -connections cannot be made compatible with the induced frame data. The frame section suggests that the reason for this is that the moduli space's connection possesses torsion. In this subsection we will generalise the  $\Xi$ -connection of [1, 7] to include torsion.

Consider a general (possibly torsional) affine connection to be a map

$$\gamma : \Gamma(TM \times TM) \rightarrow \Gamma(TM).$$

As in the constructions of the previous two sections, we'll build such a map from the twistor data via the Kodaira isomorphism. Let  $V = V^a \partial_a$  and  $W = W^a \partial_a$  be vector fields on  $M$ . We want to use the complex structure of the twistor space  $Z$  to build a torsional connection by directly constructing  $\nabla_W V = \gamma(V, W)$ .

Apply  $V$  to the patching (3) to obtain

$$V^a \partial_a \hat{w}^\mu = \mathcal{F}_\nu^\mu V^a \partial_a w^\nu \quad (9)$$

as usual. Then apply  $W$  to (9) to obtain

$$W^b \partial_b V^a \partial_a \hat{w}^\mu + W^b V^a \partial_b \partial_a \hat{w}^\mu = W^b \partial_b \mathcal{F}_\nu^\mu V^a \partial_a w^\nu + W^b \mathcal{F}_\nu^\mu \partial_b V^a \partial_a w^\nu + W^b \mathcal{F}_\nu^\mu V^a \partial_b \partial_a w^\nu.$$

As in the torsion-free case the only obstruction to this line (evaluated at  $x \in M$ ) constituting a global section of  $N_x$  is the first term on the right-hand-side, and one must decide what to do with it.

If we can write

$$\partial_b \mathcal{F}_\nu^\mu = -\hat{\rho}_{\alpha b}^\mu \mathcal{F}_\nu^\alpha + \mathcal{F}_{\beta \nu}^\mu \rho_{\nu b}^\beta \quad (10)$$

for some 0-cochain  $\{\rho\}$  of  $N_x \otimes \Lambda_x^1(M)$  for each  $x \in M$  then

$$W^b \partial_b V^a \partial_a \hat{w}^\mu + W^b V^a \partial_b \partial_a \hat{w}^\mu + W^b \hat{\rho}_{\alpha b}^\mu V^a \partial_a \hat{w}^\alpha = \mathcal{F}_\beta^\mu \left( W^b \rho_{\nu b}^\beta V^a \partial_a w^\nu + W^b \partial_b V^a \partial_a w^\beta + W^b V^a \partial_b \partial_a w^\beta \right)$$

constitutes a global section of  $N_x$  (for each  $x \in M$ ), and hence a vector field on  $M$  via the Kodaira isomorphism. One can then extract the connection symbols from

$$\Gamma_{ab}^c \partial_c w^\mu = \partial_a \partial_b w^\mu + \rho_{\nu b}^\mu \partial_a w^\nu \quad (11)$$

(or its counterpart over  $\hat{U}$ ) just as in the torsion-free case.

The connection symbols arising from (11) generically possess torsion, and the torsion-free part of the connection agrees with the torsion-free  $\Xi$ -connection of exhibited in section 1.2.1. We accordingly call the connection of this section the *torsion  $\Xi$ -connection*.

Just like the  $\Xi$ -connection the existence is determined by the non-vanishing of  $\check{H}^1(X_x, N_x \otimes N_x^*)$  and the connection is defined up to an element of  $\check{H}^0(X_x, N_x \otimes N_x^*) \otimes \Lambda_x^1(M)$  per  $x \in M$ , giving us a family of connections induced on  $M$ .

### 1.3 Newton-Cartan Geometry

The non-relativistic limit of a Lorentzian manifold is a Newton-Cartan manifold; such manifolds provide a geometrical setting for non-relativistic physics [9]. As in general relativity a four-manifold models the spacetime, and test particles travel on geodesics of a (usually torsion-free) connection. There's a metric also, though unlike in general relativity the metric and the connection are independent. In this section we will describe Newton-Cartan manifolds in some detail, taking [4] as a reference.

Before introducing the full Newton-Cartan geometry we'll begin with a subordinate definition.

**Definition 1.5.** A  $(d + 1)$ -dimensional Galilean spacetime is a triple  $(M, h, \theta)$  where

- $M$  is a  $(d + 1)$ -dimensional manifold;
- $h$  is a symmetric tensor field of valence  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  with signature  $(0 + + \dots +)$  (and so has rank  $d$ ) called the metric;
- and  $\theta$  is a closed one-form spanning the kernel of  $h$  called the clock.

The pair  $(h, \theta)$  is called a *Galilean structure*, and the number of spatial dimensions is  $d$ .

Since  $\theta$  is closed we can always locally write  $\theta = dt$  for some function  $t : M \rightarrow \mathbb{R}$ . This function is then taken as a coordinate on the time axis, a one-dimensional submanifold over which the spacetime  $M$  is fibred. We call the fibres *spatial slices* and when restricted to such a slice the metric  $h$  is a more familiar signature  $(+ \dots +)$   $d$ -metric. Throughout this paper the indices  $a, b, c$  will run from 0 to  $d$  and the spatial indices  $i, j, k$  will run from 1 to  $d$ .

**Definition 1.6.** A  $(d + 1)$ -dimensional Newton-Cartan spacetime is a quadruple  $(M, h, \theta, \nabla)$  where

- $M$  is a  $(d + 1)$ -dimensional manifold;
- $(h, \theta)$  is a Galilean structure;
- and  $\nabla$  is a torsion-free connection compatible with the Galilean structure in the sense that  $\nabla h = 0$  and  $\nabla \theta = 0$ .

Note that crucially  $\nabla$  must be specified independently of the metric and clock.

The field equations for Newton-Cartan gravity arise as the Newtonian limit of the Einstein equations [10] and are given by

$$R_{ab} = 4\pi G \rho \theta_a \theta_b \quad (12)$$

where  $R_{ab}$  is the Ricci tensor associated to  $\nabla$ ;  $G$  is Newton's constant; and  $\rho : M \rightarrow \mathbb{R}$  is the mass density. In addition to the field equations there is the Trautman condition [4]

$$h^{a[b} R^c{}_{(de)a} = 0, \quad (13)$$

where  $R^a{}_{bcd}$  is the Riemann tensor of  $\nabla$  which ensures that there exist potentials (such as the Newtonian potential) for the connection components. Newton-Cartan connections which satisfy (13) are called *Newtonian* connections.

The field equations imply that  $h$  is flat on spatial slices, and we can always introduce *Galilean* coordinates  $(t, x^i)$  such that

$$h = \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \quad \text{and} \quad \theta = dt \quad (14)$$

for  $i = 1, 2, \dots, d$ . We'll refer to (14) as the *standard* Galilean structure.

Only connections compatible with  $\theta$  and  $h$  are allowed by definition; one can show [11] that the most general such connection has components

$$\Gamma_{bc}^a = \frac{1}{2} h^{ad} (\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}) + \partial_{(b} \theta_{c)} U^a + \theta_{(b} F_{c)d} h^{ad} \quad (15)$$

where

- $U^a$  is any vector field satisfying  $\theta(U) = 1$ ;
- $F_{ab}$  is any two-form;
- and  $h_{ab}$  is uniquely determined by  $h^{ab} h_{bc} = \delta_c^a - \theta_c U^a$  and  $h_{ab} U^b = 0$ .

Possible connections, given a Galilean structure, are then determined by a choice of  $(U, F)$ . The Trautman condition (13) is equivalent to the statement that  $F$  is closed, and hence for a Newtonian connection we can locally write  $F = dA$ . As well as the obvious gauge symmetry

$$A \longmapsto A + d\chi$$

there is a further redundancy in this description. There exist *Milne boosts* which can be thought of as gauge transformations of  $(U, F)$  leaving  $\Gamma_{bc}^a$  unchanged [4].

With  $d = 3$  the most general vacuum Newton-Cartan manifold satisfying (12) and (13) then has

$$\Gamma_{tt}^i = \delta^{ij} \partial_j V \quad \text{and} \quad \Gamma_{jt}^i = \Gamma_{tj}^i = \delta_{jl} \epsilon^{ilk} \partial_k \Omega$$

where  $\delta^{ij} \partial_i \partial_j V + 2\delta^{ij} \partial_i \Omega \partial_j \Omega = 0 \quad \text{and} \quad \delta^{ij} \partial_i \partial_j \Omega = 0,$  (16)

with all other connection components vanishing. The corresponding two-form  $F$  is given by

$$F = -dV \wedge dt + \epsilon_{ijk} \delta^{kl} \partial_l \Omega dx^i \wedge dx^j.$$

The geodesic equations suggest interpreting the function  $V$  as the Newtonian (gravitational) potential and the function  $\Omega$  as a potential for generalised (spatially-varying) Coriolis forces. Note that although the degrees of freedom in a Newton-Cartan connection appear similar to that of an electromagnetic field the equations (16) governing them are more complicated.

Of recent interest has been *torsional* Newton-Cartan geometry, in which the connection is allowed to have some torsion. This is manifest in the clock failing to be closed because  $d\theta \neq 0$  is incompatible with  $\nabla\theta = 0$  for a torsion-free connection. Equation (15) is modified to include the skew part of  $\partial_a \theta_b$  too, giving rise to the torsion. Later in this paper we will construct Kodaira families with induced torsional Newton-Cartan structures, featuring clocks which aren't closed.

## 2 Examples of Frames and Connections

In this section we'll apply the results of subsections 1.1 and 1.2 in some simple cases.

### 2.1 Line Bundles on $\mathbb{P}^1$

Twistor spaces with families of submanifolds  $X_x$  having  $N_x = \mathcal{O}(k)$  for some  $k \geq 1$  are about as simple as it gets; they are, though, sufficiently sophisticated as to require some careful treatment of their canonical connections, particularly when  $k$  is odd.

Applied in these cases theorem 1.3 amounts to the construction of a *paraconformal* structure on  $M$ , i.e. a bundle isomorphism

$$TM = \odot^k \mathbb{S}'$$

as is studied in [22, 23, 24], concretely given by the frame  $e_a^{A'_1 \dots A'_k}$ .

#### 2.1.1 Odd Dimensions

When  $k$  is even the treatment is relatively straightforward: the span of the frame will give us a conformal structure and the  $\Lambda$ -connection can pick out a preferred representative.

**Theorem 2.1.** *Let  $Z \rightarrow \mathbb{P}^1$  be a complex two-fold containing a rational curve  $X_0$  with normal bundle  $N_0 = \mathcal{O}(2n)$  for some  $n \geq 1$ . The Kodaira moduli space of rational curves  $X_x$  is a  $(2n+1)$ -dimensional complex conformal manifold.*

For  $n = 1$   $Z$  is a minitwistor space [3].

#### Proof

Since  $\check{H}^1(\mathbb{P}^1, \mathcal{O}(2n)) = 0$  the rational curve  $X_0$  is a member of a  $\dim \check{H}^0(\mathbb{P}^1, \mathcal{O}(2n)) = (2n+1)$ -dimensional family of rational curves, and by theorem 1.3 we obtain a section of  $\Lambda_x^1(M) \otimes N_x$  at each point  $x \in M$  which gives rise to a frame via

$$v = e^{A'_1 \dots A'_{2n}} \pi_{A'_1} \dots \pi_{A'_{2n}}$$

and so a metric

$$g = e_{A'_1 \dots A'_{2n}} \otimes e^{A'_1 \dots A'_{2n}} \quad (17)$$

of maximal rank in the span of  $v$ . The redundancy acts as

$$e^{A'_1 \dots A'_{2n}} \mapsto \alpha e^{A'_1 \dots A'_{2n}}$$

for any  $\alpha : M \rightarrow \mathbb{C}^*$ , resulting in conformal transformations  $g \mapsto \alpha^2 g$ . □

In the case for which the patching for  $Z$  is that of  $\mathcal{O}(2n)$  (even if the sections are deformed) one may fix a particular metric from the conformal class by constructing the  $\Lambda$ -connection, which is in this case unique and exists for the  $\mathcal{O}(2n)$  patching.

We can equip  $Z$  with an involution which singles out Euclidean signature metrics. (See, for example, [17, 20].) The metric (17) is the same as that arising from the classical invariant theory described in [21].

### 2.1.2 Even Dimensions

When  $k$  is odd the situation is more complicated because the span contains no (non-degenerate) metric. In the following theorem we consider one option of what one *can* do with the frame, though this is by no means the only geometry induced on  $M$ .

**Theorem 2.2.** *Let  $Z \rightarrow \mathbb{P}^1$  be a complex two-fold containing a rational curve  $X_0$  with normal bundle  $N_0 = \mathcal{O}(2n - 1)$  for some  $n \geq 1$ . Then the Kodaira moduli space  $M$  of rational curves  $X_x$  is a  $(2n)$ -dimensional complex torsional projective manifold.*

Restricting to torsion-free connections only, for  $n = 1$  this is the standard twistor theory of projective surfaces due to Hitchin [3] and for  $n = 2$  the normal bundle  $\mathcal{O}(3)$  is that associated to exotic holonomies in the work of Bryant [24], whose twistor theory is described in terms of solutions spaces of ODEs in [22].

#### Proof

Since  $\check{H}^1(\mathbb{P}^1, \mathcal{O}(2n - 1)) = 0$  (for  $n \geq 1$ ) the rational curve  $X_0$  is a member of a  $\dim \check{H}^0(\mathbb{P}^1, \mathcal{O}(2n - 1)) = (2n)$ -dimensional family of rational curves, and by theorem 1.3 we obtain a section of  $\Lambda_x^1(M) \otimes N_x$  at each point  $x \in M$  which gives rise to a frame via

$$v = e^{A'_1 \dots A'_{2n-1}} \pi_{A'_1} \dots \pi_{A'_{2n-1}}.$$

Unlike the case of odd dimensions the span does not contain a metric of maximal rank. We can, though, construct a family of connections out of the frame. A change of global section of  $N_x \otimes N_x^*$  acts as

$$v \mapsto \alpha v \tag{18}$$

for  $\alpha : M \rightarrow \mathbb{C}^*$ , so write the frame as

$$e^{A'_1 \dots A'_{2n-1}} = \alpha(x) \varsigma_a^{A'_1 \dots A'_{2n-1}}(x) dx^a.$$

We can construct a canonical family of affine connections on  $M$  by requiring  $\nabla e^{A'_1 \dots A'_{2n-1}} = 0$ . Concretely, this gives us

$$\Gamma_{ab}^c = \varsigma_{A'_1 \dots A'_{2n-1}}^c \partial_a \varsigma_b^{A'_1 \dots A'_{2n-1}} + \delta_a^c \partial_b \ln \alpha \tag{19}$$

where  $\varsigma_{A'_1 \dots A'_{2n-1}}^a$  is the inverse of  $\varsigma_a^{A'_1 \dots A'_{2n-1}}$ . The connections described in (19) possess torsion whenever  $de^{A'_1 \dots A'_{2n-1}} \neq 0$ ; their torsion-free parts (and hence their geodesics) constitute a projective structure, in that a change of  $\alpha$  leaves the unparametrised geodesics unaltered.  $\square$

Now consider the canonical connections induced on  $M$  without reference to the frame section. We have  $N_x \otimes N_x^* = \mathcal{O}$ , so the torsion  $\Xi$ -connection always exists and depends on a single one-form on  $M$ . In the case  $Z = \mathcal{O}(1)$  the torsion-free  $\Xi$ -connection is a standard flat projective structure.

On the other hand we have

$$N_x \otimes (N_x^* \odot N_x^*) = \mathcal{O}(1 - 2n)$$

so

$$\check{H}^0(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)) = 0$$

and

$$\check{H}^1(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)) = \mathbb{C}^{2n-2}.$$

Thus the  $\Lambda$ -connection, when it exists, is unique.

For  $Z = \mathcal{O}(1)$  we find that  $\Gamma_{bc}^a = 0$ , so the moduli space comes equipped with a preferred representative of the projective structure, and moreover one which is metrisable. There is thus in this case an important corollary:  $M$  is equipped with a flat metric  $h_{ab}$ . We simply impose  $\nabla h = 0$  and the torsion-free condition (by analogy with the existence of the Levi-Civita connection), giving us a metric with constant coefficients (which is unique up to diffeomorphisms in two dimensions). This will be important for theorem 3.1. (Note that this does not imply that all such  $\Lambda$ -connections give rise to metrics: the connection is not guaranteed to be metrisable.)

In theorem 2.2 we chose to make the whole frame parallel, but we had other options. Another would be to construct a family of connections by declaring the form  $e_{A'_1 \dots A'_{2n-1}} \otimes e^{A'_1 \dots A'_{2n-1}}$  to be parallel. In two dimensions this form is complex symplectic, and the connection is known as a symplectic connection.

## 2.2 $\mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1) \rightarrow \mathbb{P}^1$

**Theorem 2.3.** *Let  $Z \rightarrow \mathbb{P}^1$  be a complex three-fold containing a rational curve  $X_0$  with normal bundle  $N_0 = \mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1)$  for some  $n \geq 1$ . Then the Kodaira moduli space of rational curves  $X_x$  is a  $(4n)$ -dimensional complexified conformal manifold.*

For  $n = 1$   $Z$  is a standard twistor space, and in a different context this class of normal bundles is the setting for the heavenly hierarchy described in [25].

Note that theorem 2.3 is a different construction of  $4n$ -dimensional moduli spaces to that in [14, 15], where the authors induce quaternionic structures on Kodaira families of global sections of manifolds with normal bundle  $\oplus^{2n} \mathcal{O}(1)$ .

### Proof

Since

$$\check{H}^1(\mathbb{P}^1, \mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1)) = 0$$

the rational curve  $X_0$  is a member of a family of dimension

$$\dim \check{H}^0(\mathbb{P}^1, \mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1)) = 4n,$$

and by theorem 1.3 we obtain a section of  $\Lambda_x^1(M) \otimes N_x$  at each point  $x \in M$  which gives rise to a frame  $e^{AA'_1 \dots A'_{2n-1}}$  via

$$v^A = e^{AA'_1 \dots A'_{2n-1}} \pi_{A'_1} \dots \pi_{A'_{2n-1}}$$

and so a metric

$$g = e_{AA'_1 \dots A'_{2n-1}} \otimes e^{AA'_1 \dots A'_{2n-1}}.$$

The redundancy acts via an invertible global section of  $N_x \otimes N_x^*$ , which takes

$$\begin{pmatrix} e^{00' \dots 0' \lambda^{2n-1}} + e^{00' \dots 0' 1' \lambda^{2n-2}} + \dots + e^{01' \dots 1' \lambda^{2n-1}} \\ e^{10' \dots 0' \lambda^{2n-1}} + e^{10' \dots 0' 1' \lambda^{2n-2}} + \dots + e^{11' \dots 1' \lambda^{2n-1}} \end{pmatrix} \mapsto \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} e^{00' \dots 0' \lambda^{2n-1}} + e^{00' \dots 0' 1' \lambda^{2n-2}} + \dots + e^{01' \dots 1' \lambda^{2n-1}} \\ e^{10' \dots 0' \lambda^{2n-1}} + e^{10' \dots 0' 1' \lambda^{2n-2}} + \dots + e^{11' \dots 1' \lambda^{2n-1}} \end{pmatrix}$$

for any four functions  $\phi_B^A = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} : \mathbb{C} \rightarrow \text{GL}(2, \mathbb{C})$ , resulting in

$$e^{AA'_1 \dots A'_{2n-1}} \mapsto \phi_B^A e^{BA'_1 \dots A'_{2n-1}}$$

and so

$$g \mapsto \epsilon_{AD} \phi_B^D \phi_C^A e^{BA'_1 \dots A'_{2n-1}} \otimes e^{CA'_1 \dots A'_{2n-1}} = (\det \phi) g.$$

We thus obtain a conformal transformation.  $\square$

Again the  $\Lambda$ -connection, when it exists, can be used to fix a particular representative  $g \in [g]$  by imposing  $\nabla g = 0$ . (The  $\Lambda$ -connection exists in this case when the deformation giving rise to  $Z$  does not fall within  $\check{H}^1(X_x, N_x \otimes (N_x^* \odot N_x^*))$  when restricted to twistor lines.)

## 2.3 Limits in $4n$ Dimensions

Let  $D = 4n$  where  $n > 0$  is an integer.

**Theorem 2.4.** *Let  $Z_c \rightarrow \mathbb{P}^1$  be a one-parameter family of vector bundles with patching*

$$\hat{T} = T + \frac{S}{c\lambda^{2n-1}}$$

$$\hat{S} = \lambda^{2-D} S.$$

For  $c \neq \infty$  the normal bundle to all rational curves  $X_x$  is  $N_x = \mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1)$  and the homogeneous frame section is

$$v = \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} \quad \text{where} \quad v^A = e^{AA'_1 \dots A'_{2n-1}} \pi_{A'_1 \dots A'_{2n-1}}$$

giving rise to a non-degenerate metric

$$g = e_{AA'_1 \dots A'_{2n-1}} \otimes e^{AA'_1 \dots A'_{2n-1}}$$

on  $M$ . For  $c = \infty$  the normal bundle to all rational curves  $X_x$  is  $N_x = \mathcal{O} \oplus \mathcal{O}(4n-2)$  and the homogeneous frame section is

$$v = \begin{pmatrix} \theta \\ e^{A'_1 \dots A'_{4n-2}} \pi_{A'_1} \dots \pi_{A'_{4n-2}} \end{pmatrix}$$

giving rise to a Galilean structure with clock  $\theta$  and

$$h^{-1} = e_{A'_1 \dots A'_{4n-2}} \otimes e^{A'_1 \dots A'_{4n-2}}.$$

The induced geometry is subject to a redundancy, which in the  $c \neq \infty$  case amounts to a conformal ambiguity and in the  $c = \infty$  constitutes the non-metric nature of the connection's gravitational sector. For  $n = 1$  this is the standard Newtonian limit of twistor theory presented in [2], and for  $c = \infty$  the manifold is a Newton-Cartan manifold with arbitrary gravitational sector.

## Proof

We need to begin by identifying the isomorphism class of  $N_x$ , which will be the same for all  $x \in M$  because  $Z_c$  is the total space of a vector bundle. For  $c = \infty$  the patching for  $N_x$  is

$$\mathcal{F} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{2-D} \end{pmatrix}$$

and so the isomorphism class is obvious:  $N_x = \mathcal{O} \oplus \mathcal{O}(D-2)$ . We thus obtain a frame section

$$v = H^{-1} \begin{pmatrix} dT| \\ dS| \end{pmatrix}$$

where  $H^{-1}$  is a general global section of  $N_x \otimes N_x^*$ . The rational curves are given by

$$T| = t \quad S| = x_0 + x_1\lambda + \dots + x_{D-2}\lambda^{D-2}$$

and span of  $v$  is the Galilean structure which was advertised.

For  $c \neq \infty$  the patching of  $N_x$  splits as

$$\mathcal{F} = \hat{H} \begin{pmatrix} \lambda^{1-2n} & 0 \\ 0 & \lambda^{1-2n} \end{pmatrix} H^{-1}$$

where (for instance)

$$H = \begin{pmatrix} 1 & 0 \\ -c\lambda^{2n-1} & 1 \end{pmatrix} \quad \text{and} \quad \hat{H} = \begin{pmatrix} 0 & c^{-1} \\ -c & \hat{\lambda}^{2n-1} \end{pmatrix}.$$

This exhibits the normal bundle's isomorphism class as  $\mathcal{O}(2n-1) \oplus \mathcal{O}(2n-1)$ , and the frame section is

$$v = H_0 \begin{pmatrix} 1 & 0 \\ c\lambda^{2n-1} & 1 \end{pmatrix} \begin{pmatrix} dT| \\ dS| \end{pmatrix} = H_0 \begin{pmatrix} dT| \\ dS| + c\lambda^{2n-1}dT| \end{pmatrix}$$

for an arbitrary non-degenerate matrix of functions  $H_0$ . The rational curves are given by

$$T| = t - \frac{1}{c} \sum_{m=0}^{2-2n} x_{m+2n} \lambda^{1+m}$$

$$S| = x_0 + x_1\lambda + \dots + x_{D-2}\lambda^{D-2},$$

which results in a frame section of the advertised form. □



### 3 Newtonian Twistor Theory in Three Dimensions

The twistor theory of complexified three-dimensional manifolds with non-degenerate connections is called *minitwistor* theory and is well understood [3]. In this section we will consider the twistor theory of three-dimensional Newton-Cartan manifolds. The relevant twistor spaces are three-dimensional and will be characterised by the normal bundle to twistor lines  $X_x$  being

$$N_x = \mathcal{O} \oplus \mathcal{O}(1).$$

Families of rational curves with normal bundles isomorphic to  $\mathcal{O} \oplus \mathcal{O}(1)$  have been considered in [26], where it is described that the tangent spaces of the three-dimensional moduli space come equipped with preferred one-parameter families of null rays. As we shall see below, this makes the isomorphism class  $\mathcal{O} \oplus \mathcal{O}(1)$  well-suited to describing Newton-Cartan structures in three dimensions.

It is straightforward to see that

$$\check{H}^1(\mathbb{P}^1, N_x) = 0 \quad \text{and} \quad \check{H}^0(\mathbb{P}^1, N_x) = \mathbb{C}^3$$

and so a three-dimensional moduli space is feasible. Additionally

$$\check{H}^1(\mathbb{P}^1, N_x \otimes N_x^*) = 0 \quad \text{and} \quad \check{H}^0(\mathbb{P}^1, N_x \otimes N_x^*) = \mathbb{C}^4$$

so that the isomorphism class is stable; the  $\Xi$ -connection always exists and always depends on four arbitrary one-forms. Finally we see that

$$\check{H}^1(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)) = \mathbb{C} \quad \text{and} \quad \check{H}^0(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)) = \mathbb{C}^4$$

so that the  $\Lambda$ -connection fails to exist for some Kodaira deformations, and when it does exist it is not unique, depending on four arbitrary functions. The case in which the  $\Lambda$ -connection fails to exist is when the deformation introduces torsion.

We'll begin by considering the undeformed case  $Z = \mathcal{O} \oplus \mathcal{O}(1)$ . After calculating the canonical connections and the Galilean structure we will then proceed to deform  $Z$ .

#### 3.1 Galilean Structures and Canonical Connections

In this subsection we'll discuss the canonical geometry induced on the moduli space of global sections of  $Z = \mathcal{O} \oplus \mathcal{O}(1)$ . The patching is

$$\hat{T} = T \quad \hat{\Omega} = \lambda^{-1} \Omega$$

where  $\hat{\lambda} = \lambda^{-1}$  as usual for the base  $\mathbb{P}^1$ . The global sections are

$$w^\mu| = \begin{pmatrix} T| \\ \Omega| \end{pmatrix} = \begin{pmatrix} t \\ y + z\lambda \end{pmatrix}$$

for  $x^a = (t, y, z) \in \mathbb{C}^3 = M$ . (Recall that a vertical slash indicates the restriction to rational curves.) We'll also wish to use homogeneous coordinates  $[\pi_{A'}]$  on the base and  $\omega$  for the  $\mathcal{O}(1)$  fibre, writing

$$\omega| = x^{A'} \pi_{A'}$$

for the global sections.

**Theorem 3.1.** *Let  $Z = \mathcal{O} \oplus \mathcal{O}(1)$  with global sections  $X_x$ . The moduli space  $M \ni x$  of these rational curves is a complex three-dimensional manifold equipped with a family of Newton-Cartan structures parametrised by three arbitrary functions on  $M$  and an element of  $GL(2, \mathbb{C})$ .*

Two of these functions determine the gravitational field; the other is a conformal factor.

**Proof**

To begin we will construct the frame induced on  $M$  using theorem 1.3, which is done by splitting the patching  $\mathcal{F}$  for the normal bundle; we must solve

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} H = \hat{H} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for

$$H = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \quad \text{and} \quad \hat{H} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix}$$

as holomorphic maps from  $U$  and  $\hat{U}$  to  $GL(2, \mathbb{C})$ . This amounts to the four individual splitting problems

$$\begin{aligned} \hat{h}_1 &= h_1 & \hat{h}_2 &= \lambda h_2 \\ \hat{h}_3 &= \lambda^{-1} h_3 & \hat{h}_4 &= h_4 \end{aligned}$$

whose general solution is

$$H = \begin{pmatrix} m & 0 \\ a_0 + a_1 \lambda & k \end{pmatrix}$$

for four arbitrary holomorphic functions  $(m, k, a_0, a_1)$  on  $M$  constrained only by  $m \neq 0$  and  $k \neq 0$ . The frame section can then be read off from  $v = H^{-1}dw$ , giving us a clock

$$\theta = m^{-1}dt \tag{20}$$

and spatial one-forms

$$\begin{aligned} e^{0'} &= k^{-1} \left( dx^{0'} - m^{-1} a_1 dt \right) \\ e^{1'} &= k^{-1} \left( dx^{1'} - m^{-1} a_0 dt \right). \end{aligned}$$

To proceed further we must calculate the  $\Lambda$ -connection on  $M$  as described in section 1.1. The splitting problem to be solved is

$$0 = -\hat{\sigma}_{\nu\rho}^\mu \mathcal{F}_\alpha^\nu \mathcal{F}_\beta^\rho + \mathcal{F}_\nu^\mu \sigma_{\alpha\beta}^\nu$$

for a 0-cochain  $\{\sigma\}$  valued in  $N_x \otimes (N_x^* \odot N_x^*)$  on each twistor line  $X_x$ , which amounts to

$$\begin{aligned} \hat{\sigma}_{TT}^T &= \sigma_{TT}^T & \hat{\sigma}_{T\Omega}^T &= \lambda \sigma_{T\Omega}^T & \hat{\sigma}_{\Omega\Omega}^T &= \lambda^2 \sigma_{\Omega\Omega}^T \\ \hat{\sigma}_{TT}^\Omega &= \lambda^{-1} \sigma_{TT}^\Omega & \hat{\sigma}_{T\Omega}^\Omega &= \sigma_{T\Omega}^\Omega & \hat{\sigma}_{\Omega\Omega}^\Omega &= \lambda \sigma_{\Omega\Omega}^\Omega \end{aligned}$$

and so

$$\sigma_{TT}^T = \Sigma \quad \sigma_{TQ}^T = 0$$

$$\begin{aligned}\sigma_{\Omega\Omega}^T &= 0 & \sigma_{TT}^\Omega &= \phi_0 + \lambda\phi_1 \\ \sigma_{T\Omega}^\Omega &= \chi & \sigma_{\Omega\Omega}^\Omega &= 0\end{aligned}$$

for any four functions  $(\Sigma, \chi, \phi_0, \phi_1)$  on  $M$ . The connection symbols are then

$$\begin{aligned}\Gamma_{tt}^t &= \Sigma & \Gamma_{it}^t &= 0 & \Gamma_{ij}^t &= 0 \\ \Gamma_{tt}^y &= \phi_0 & \Gamma_{tt}^z &= \phi_1 \\ \Gamma_{yt}^y &= \chi & \Gamma_{zt}^y &= 0 & \Gamma_{yt}^z &= 0 & \Gamma_{zt}^z &= \chi \\ \Gamma_{jk}^i &= 0.\end{aligned}$$

The result is that the moduli space comes equipped with a family of connections containing gravitational forces (described by the functions  $\phi_0$  and  $\phi_1$ ).

The connection allows us to restrict the clock (20) by imposing  $\nabla\theta = 0$ , which tells us that

$$\Sigma = -\partial_t \ln m$$

and

$$\partial_{A'} m = 0,$$

so that  $m$  is a function of  $t$  alone. Given that  $m$  is now just a non-vanishing function of  $t$ , we see that (20) is a standard Newton-Cartan clock, where the function  $m$  just allows for diffeomorphisms of the time axis, with  $\Sigma$  ensuring that upon such diffeomorphisms the clock remains parallel.

To complete the proof we must construct a family of Newton-Cartan metrics. The data already induced defines a metric as follows, by requiring the usual conditions  $h(\theta, \cdot) = 0$  and  $\nabla h = 0$ . These imply that we must have  $h^{at} = 0$  and that  $h^{ij}$  obeys

$$\partial_t h^{ij} + 2\chi h^{ij} = 0$$

and

$$\partial_k h^{ij} = 0.$$

We deduce that  $h^{ij}$  must be any element of  $\text{GL}(2, \mathbb{C})$  multiplied by an arbitrary non-vanishing function of  $t$ , and we also have that  $\chi = \chi(t)$  only. Constant non-degenerate two-by-two metrics are all equal up to (restricted) diffeomorphisms

$$y \mapsto \alpha y + \beta z \quad z \mapsto \gamma y + \delta z$$

for  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}(2, \mathbb{C})$ , so we are free to take any such member  $\tilde{h}^{ij}$  as our metric, giving us

$$h^{ij} = \kappa(t) \tilde{h}^{ij}$$

where

$$\kappa = \exp \left\{ -2 \int \chi dt \right\}$$

is non-vanishing and determined by the arbitrary function  $\chi$ .

Thus we have a family of Galilean structures  $(h, \theta)$  and a family of connections  $\nabla$ , two of whose arbitrary functions describe gravitational forces, completing the proof.  $\square$

To fix a specific gravitational sector for the Newton-Cartan manifold we require, as in [2], some additional data on  $Z$ . The following theorem provides one way of specifying this data.

**Theorem 3.2.** Equip  $Z = \mathcal{O} \oplus \mathcal{O}(1)$  with a 1-cocycle taking values in its canonical bundle  $K \rightarrow Z$ . This induces a preferred global section of  $\mathcal{O}(1)$  which can be used to fix a complex Newton-Cartan structure out of the complex Newton-Cartan structure of theorem 3.1, where the gravitational sector is locally of the form  $\Gamma_{tt}^i = g^i$  for  $\mathbf{g}$  divergence-free and determined uniquely by  $f$ .

### Proof

A simple calculation shows that  $K = \mathcal{O}(-3)_Z$ , and so a 1-cocycle is represented by a function  $f$  of weight minus three in homogeneous coordinates, and provides a (Serre-dual) global section of  $\mathcal{O}(1)$  by

$$\phi_{A'}(x) = \frac{1}{2\pi i} \oint_{\Gamma} \pi_{A'} f(|T|, \omega|, \pi_{B'}) \pi \cdot d\pi$$

where  $\omega| = x^{A'} \pi_{A'} = y \pi_{1'} + z \pi_{0'}$ . We have

$$\frac{\partial}{\partial x^{A'}} \phi^{A'} = 0$$

automatically. Taking this to fix  $\sigma_{TT}^\Omega$  we have

$$\Gamma_{tt}^{A'} \partial_{A'} \omega| = \phi^{A'} \pi_{A'}$$

and so

$$\Gamma_{tt}^y = \phi^{1'} \quad \Gamma_{tt}^z = \phi^{0'}.$$

Thus the gravitational sector is fixed to be a unique divergence-free  $\mathbf{g}$  given by the global  $\phi^{A'}$ . We note that the divergence-free condition ensures that the Newton-Cartan spacetime is vacuum.  $\square$

### Torsion-Free $\Xi$ -Connection

Theorem 3.1 employed the  $\Lambda$ -connection because it is the most powerful construction available in terms of constraining moduli space geometry. It is interesting, however, to consider the  $\Xi$ -connection also so as to compare the flat model of theorem 3.1 with the torsion-inducing deformations of theorem 3.3 where the torsion  $\Xi$ -connection is the only connection available.

Here we'll calculate the canonical torsion-free  $\Xi$ -connection for  $Z = \mathcal{O} \oplus \mathcal{O}(1)$ . The calculation is straightforward; since  $\partial_a \mathcal{F}_\nu^\mu = 0$  one must solve

$$0 = -\hat{\chi}_{\nu a}^\mu \mathcal{F}_\rho^\nu + \mathcal{F}_\nu^\mu \chi_{\rho a}^\nu$$

for the most general 0-cochain  $\{\chi_{\nu a}^\mu\}$  of  $N_x \otimes N_x^* \otimes \Lambda_x^1(M)$  for each  $x \in M$ , where

$$\mathcal{F}_\nu^\mu = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

This becomes the four individual splitting problems given by

$$\begin{aligned} \hat{\chi}_{T a}^T &= \chi_{T a}^T & \hat{\chi}_{\Omega a}^T &= \lambda \chi_{\Omega a}^T \\ \hat{\chi}_{T a}^\Omega &= \lambda^{-1} \chi_{T a}^\Omega & \hat{\chi}_{\Omega a}^\Omega &= \chi_{\Omega a}^\Omega, \end{aligned}$$

which have the general solution

$$\chi_{T\ a}^T = C_a \quad \chi_{\Omega\ a}^T = 0$$

$$\chi_{T\ a}^\Omega = A_a^0 + \lambda A_a^1 \quad \chi_{\Omega\ a}^\Omega = B_a$$

for arbitrary one-forms  $(A, B, C, \kappa)$  on  $M$ . We then read-off the connection from

$$\Gamma_{bc}^a \partial_a w^\mu = \partial_b \partial_c w^\mu + \chi_{\nu\ b}^\mu \partial_c w^\nu + \chi_{\nu\ c}^\mu \partial_b w^\nu$$

which here leads to

$$\begin{aligned} \Gamma_{tt}^t &= 2C_t & \Gamma_{it}^t &= C_i & \Gamma_{ij}^t &= 0 \\ \Gamma_{tt}^y &= A_t^0 & \Gamma_{tt}^z &= A_t^1 \\ \Gamma_{yt}^y &= A_y^0 + B_t & \Gamma_{zt}^y &= A_z^0 & \Gamma_{zt}^z &= A_z^1 + B_t & \Gamma_{yt}^z &= A_y^1 \\ \Gamma_{ij}^k &= 2B_{(i} \delta_{j)}^k. \end{aligned}$$

Thus the  $\Xi$ -connection comprises all connections which have  $\Gamma_{ij}^t = 0$  and have flat projective structures as their spatial sectors. Compatibility with the (closed) clock  $\theta = m^{-1}dt$  then imposes

$$\Gamma_{ab}^t = \delta_a^t m \partial_b (m^{-1})$$

so (recalling that the connection is torsion-free) one must put  $C_i = 0$  and

$$C_t = \frac{1}{2} m \partial_t (m^{-1}).$$

The remaining freedom in  $(\Gamma_{ab}^c, \theta_a)$  is then given by three one-forms and one non-vanishing function on the time axis.

### 3.2 Deformations and Torsion

A natural next step is to consider deforming the complex structure of  $Z = \mathcal{O} \oplus \mathcal{O}(1)$ , and a case of interest is when we write  $\mathcal{O} \oplus \mathcal{O}(1)$  as a trivial affine line bundle on  $\mathcal{O}(1)$  and then deform the patching for the affine line bundle so that we have

$$\hat{T} = T + f(\Omega, \lambda) \tag{21}$$

$$\hat{\Omega} = \lambda^{-1} \Omega$$

as the patching for  $Z \rightarrow \mathcal{O}(1) \rightarrow \mathbb{P}^1$ . The analogous deformation in four-dimensional Newtonian twistor theory [2] leads to a jump in the isomorphism class of the normal bundle to every<sup>1</sup> twistor line from  $\mathcal{O} \oplus \mathcal{O}(2)$  to  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ . In the three-dimensional case this cannot be what occurs, because the isomorphism class of the normal bundle is stable.

The deformation leading to (21), when restricted to twistor lines, corresponds exactly to the part of  $N_x \otimes (N_x^* \odot N_x^*)$  which causes  $\tilde{H}^1(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*))$  to fail to vanish. Thus we expect something to go “wrong” with the torsion-free affine connection on  $M$ ; in fact what happens is that the connection fails to be torsion-free.

---

<sup>1</sup>Almost every twistor line; see section 3.5.

**Theorem 3.3.** *Let  $Z$  be the total space of an affine line bundle on  $\mathcal{O}(1)$  with trivial underlying translation bundle whose patching is*

$$\hat{T} = T + f$$

*where  $f$  represents a cohomology class in  $\check{H}^1(\mathcal{O}(1), \mathcal{O}_{\mathcal{O}(1)})$ . The three-parameter family of global sections  $X_x$  have normal bundle  $X_x = \mathcal{O} \oplus \mathcal{O}(1)$  and the moduli space  $M$  of those sections is a complex three-dimensional manifold equipped with a family of torsional Newton-Cartan structures parametrised by two arbitrary one-forms on  $M$ , two functions on  $M$ , and an element of  $GL(2, \mathbb{C})$ .*

### Proof

The proof will proceed in stages.

1. Using theorem 1.3 we will construct a clock one-form  $\theta$  on  $M$  depending on one non-vanishing function  $m$  on  $M$ ; this clock will not be closed, meaning that it cannot be made compatible with a torsion-free connection.
2. We will then construct the torsion  $\Xi$ -connection described in section 1.2.3; its connection symbols will depend on four arbitrary one-forms  $(A^0, A^1, B, C)$  on  $M$ .
3. Imposing  $\nabla\theta = 0$  for the torsion  $\Xi$ -connection is then possible, and results in the fixing of  $C$ .
4. The remaining piece of data, the Newton-Cartan metric  $h$ , will then be constructed by imposing  $h(\theta, \cdot) = 0$  and  $\nabla h = 0$ . This restricts the remaining one-forms by a closure condition and determines the metric up to a choice of constant two-by-two non-degenerate matrix  $\tilde{h}$ .

To begin we must construct the twistor functions by finding the global sections of  $Z \rightarrow \mathbb{P}^1$ , as these are required for theorem 1.3. It'll be useful to expand the representative  $f$  when restricted to sections of  $\mathcal{O}(1)$ . For  $|\Omega\rangle = y + z\lambda$  (with  $y$  and  $z$  coordinates on  $M$ ) we can write

$$f(|\Omega\rangle, \lambda) = \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n$$

where  $\gamma_n(y, z)$  are functions one can extract via integration.

We then have

$$\hat{T} = T + \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n$$

and so the twistor functions are

$$T| = t - \sum_{n=1}^{\infty} \gamma_n \lambda^n \quad \hat{T}| = t + \sum_{n=-\infty}^0 \gamma_n \lambda^n$$

where we have chosen to put the  $\gamma_0$  term into  $\hat{T}|$  (without loss of generality: we could always effect the diffeomorphism  $t \mapsto t - \gamma_0$ ). Stage one of the proof is then to use these twistor functions to calculate a frame section via theorem 1.3. This involves solving the splitting problem

$$\begin{pmatrix} 1 & \frac{\partial f}{\partial \Omega} \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where as usual

$$H = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \quad \text{and} \quad \hat{H} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix}$$

constitute holomorphic maps to  $\text{GL}(2, \mathbb{C})$  from  $U$  and  $\hat{U}$  respectively for each global section  $X_x$ . Expand the first derivative of  $f$  in a similar fashion to above, putting

$$\left. \frac{\partial f}{\partial \Omega} \right| = \sum_{n=-\infty}^{\infty} \phi_n \lambda^n \quad (22)$$

for functions  $\phi_n(y, z)$ . We then have

$$\hat{h}_3 = \lambda^{-1} h_3 \quad \Rightarrow \quad h_3 = a + b\lambda$$

and

$$\hat{h}_4 = h_4 \quad \Rightarrow \quad h_4 = e$$

for functions  $(a, b, e)$  on  $M$ . The other two components of the splitting problem are then

$$\begin{aligned} \hat{h}_2 &= \lambda h_2 + \lambda \left( \sum_{n=-\infty}^{\infty} \phi_n \lambda^n \right) e \\ \Rightarrow \quad h_2 &= -e \sum_{n=0}^{\infty} \phi_n \lambda^n \end{aligned}$$

and

$$\begin{aligned} \hat{h}_1 &= h_1 + \left( \sum_{n=-\infty}^{\infty} \phi_n \lambda^n \right) (a + b\lambda) \\ \Rightarrow \quad h_1 &= m - \left( \sum_{n=0}^{\infty} \phi_n \lambda^n \right) (a + b\lambda) \end{aligned}$$

where  $m$  is a new function on  $M$  parametrising the non-uniqueness in the splitting. We have  $\det H = em$  so we must impose  $e \neq 0$  and  $m \neq 0$ . The frame section is then

$$\begin{aligned} v = H^{-1} \begin{pmatrix} dT \\ d\Omega \end{pmatrix} &= \frac{1}{em} \begin{pmatrix} e & e \sum_{n=0}^{\infty} \phi_n \lambda^n \\ -(a + b\lambda) & m - (\sum_{n=0}^{\infty} \phi_n \lambda^n)(a + b\lambda) \end{pmatrix} \begin{pmatrix} dt - \sum_{n=1}^{\infty} d\gamma_n \lambda^n \\ dy + \lambda dz \end{pmatrix} \\ \Rightarrow \quad v &= \frac{1}{em} \begin{pmatrix} e(dt + \phi_0 dy) \\ m(dy + \lambda dz) - (a + b\lambda)(dt + \phi_0 dy) \end{pmatrix} \end{aligned}$$

and the clock can be read off:

$$\theta = m^{-1} (dt + \phi_0 dy). \quad (23)$$

Recall that  $\phi_0 = \phi_0(y, z)$  and so for any choice of  $m \neq 0$  we have that  $d\theta \neq 0$ , suggesting that the moduli space possesses Newton-Cartan torsion. The clock (23) cannot be made compatible with any torsion-free connection as we must have

$$\begin{aligned} \nabla_b \theta_a &= \partial_b \theta_a - \Gamma_{ab}^c \theta_c = 0 \\ \Rightarrow \quad \Gamma_{ab}^t &= m \partial_b \theta_a - \Gamma_{ab}^y \phi_0. \end{aligned} \quad (24)$$

If  $\Gamma_{ab}^c$  were torsion-free then skew-symmetrising over  $ab$  in (24) would give  $d\theta = 0$ , which is never true for this class of one-forms.

In stage two of the proof we construct the torsion  $\Xi$ -connection of section 1.2.3 with respect to which the clock can be made parallel. We must solve the splitting problem

$$\partial_b \mathcal{F}_\nu^\mu = -\hat{\rho}_{\alpha b}^\mu \mathcal{F}_\nu^\alpha + \mathcal{F}_\beta^\mu \rho_{\nu b}^\beta. \quad (25)$$

in the case

$$\partial_b \mathcal{F}_\nu^\mu = \partial_b \begin{pmatrix} 1 & \frac{\partial f}{\partial \Omega} \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial^2 f}{\partial \Omega^2} | (\delta_b^y + \delta_b^z \lambda) \\ 0 & 0 \end{pmatrix}.$$

Equation (25) constitutes four couple splitting problems; we must find the global sections of the following four patchings:

$$\hat{\rho}_{Tb}^T = \rho_{Tb}^T + \frac{\partial f}{\partial \Omega} | \rho_{Tb}^\Omega; \quad (26)$$

$$\hat{\rho}_{\Omega b}^T = \lambda \rho_{\Omega b}^T + \lambda \frac{\partial f}{\partial \Omega} | (\rho_{\Omega b}^\Omega - \rho_{Tb}^T) - \lambda \frac{\partial^2 f}{\partial \Omega^2} | (\delta_b^y + \delta_b^z \lambda); \quad (27)$$

$$\hat{\rho}_{Tb}^\Omega = \lambda^{-1} \rho_{Tb}^\Omega; \quad (28)$$

$$\hat{\rho}_{\Omega b}^\Omega = \rho_{\Omega b}^\Omega - \lambda \hat{\rho}_{Tb}^\Omega \frac{\partial f}{\partial \Omega} |. \quad (29)$$

Equations (26), (28), and (29) are immediately tractable if we again make use of the expansion (22). Their most general global sections are given by

$$\begin{aligned} \rho_{Tb}^\Omega &= A_b^0 + A_b^1 \lambda & \hat{\rho}_{Tb}^\Omega &= \hat{\lambda} A_b^0 + A_b^1 \\ \rho_{Tb}^T &= C_b - \sum_{n=0}^{\infty} \phi_n \lambda^n A_b^0 - \sum_{n=-1}^{\infty} \phi_n \lambda^{n+1} A_b^1 \\ \hat{\rho}_{Tb}^T &= C_b + \sum_{n=1}^{\infty} \phi_{-n} \hat{\lambda}^n A_b^0 + \sum_{n=2}^{\infty} \phi_{-n} \hat{\lambda}^{n-1} A_b^1 \\ \rho_{\Omega b}^\Omega &= B_b + \sum_{n=1}^{\infty} \phi_n \lambda^n A_b^0 + \sum_{n=0}^{\infty} \phi_n \lambda^{n+1} A_b^1 \\ \hat{\rho}_{\Omega b}^\Omega &= B_b - \sum_{n=0}^{\infty} \phi_{-n} \hat{\lambda}^n A_b^0 - \sum_{n=1}^{\infty} \phi_{-n} \hat{\lambda}^{n-1} A_b^1 \end{aligned}$$

where  $(A^0, A^1, B, C)$  are arbitrary one-forms on  $M$  carrying the non-uniqueness in the splitting. The remaining equation (27) is, after a little work, given by

$$\begin{aligned} \hat{\rho}_{\Omega b}^T &= \lambda \rho_{\Omega b}^T + \sum_{m=-\infty}^{\infty} \phi_m \lambda^{m+1} (B_b - C_b) \\ &+ \sum_{m=-\infty}^{\infty} \left( \left[ \sum_{n=1}^{\infty} - \sum_{n=-\infty}^{-1} \right] \phi_n \phi_m \lambda^{n+m+1} A_b^0 + \left[ \sum_{n=0}^{\infty} - \sum_{n=-\infty}^{-2} \right] \phi_n \phi_m \lambda^{m+n+2} A_b^1 \right) \\ &- \lambda \frac{\partial^2 f}{\partial \Omega^2} | (\delta_b^y + \delta_b^z \lambda). \end{aligned}$$



This equation is the patching for an affine line bundle on  $\mathbb{P}^1$  with underlying translation bundle  $\mathcal{O}(-1)$  (for each direction  $x^b$  on  $M$ ) and hence always has a unique solution.

If we expand the second derivative of  $f$  such that

$$\left. \frac{\partial^2 f}{\partial \Omega^2} \right| = \sum_{m=-\infty}^{\infty} \psi_m \lambda^m$$

then (after a calculation) we can write the solution to the splitting problem as

$$\begin{aligned} \rho_{\Omega b}^T &= - \sum_{m=0}^{\infty} \phi_m \lambda^m (B_b - C_b) - \sum_{k=1}^{\infty} \lambda^{k-1} \mathcal{W}_{kb} + \sum_{m=0}^{\infty} (\delta_b^y \psi_m + \delta_b^z \psi_{m-1}) \lambda^m \\ \hat{\rho}_{\Omega b}^T &= \sum_{m=1}^{\infty} \phi_{-m} \hat{\lambda}^{m-1} (B_b - C_b) + \sum_{k=0}^{\infty} \hat{\lambda}^k \mathcal{W}_{-kb} - \sum_{m=1}^{\infty} (\delta_b^y \psi_{-m} + \delta_b^z \psi_{-(m+1)}) \hat{\lambda}^{m-1} \end{aligned}$$

where for convenience we define

$$\mathcal{W}_{kb} = \sum_{n=1}^{\infty} [A_b^0 (\phi_n \phi_{k-1-n} - \phi_{-n} \phi_{k-1+n}) + A_b^1 (\phi_n \phi_{k-2-n} - \phi_{-n} \phi_{k-2+n})] + A_b^1 (\phi_0 \phi_{k-2} + \phi_{-1} \phi_{k-1}).$$

Having solved the splitting problem it is straightforward to extract  $\Gamma_{ab}^c$  from (11); we have

$$\Gamma_{ab}^c \left( \delta_c^t - \sum_{n=1}^{\infty} (\partial_c \gamma_n) \lambda^n \right) = - \sum_{n=1}^{\infty} (\partial_a \partial_b \gamma_n) \lambda^n + \rho_{Tb}^T \left( \delta_a^t - \sum_{n=1}^{\infty} (\partial_a \gamma_n) \lambda^n \right) + \rho_{\Omega b}^T (\delta_a^y + \delta_a^z \lambda)$$

and

$$\Gamma_{ab}^c (\delta_c^y + \delta_c^z \lambda) = \rho_{Tb}^{\Omega} \left( \delta_a^t - \sum_{n=1}^{\infty} (\partial_a \gamma_n) \lambda^n \right) + \rho_{\Omega b}^{\Omega} (\delta_a^y + \delta_a^z \lambda)$$

from which we can read off

$$\begin{aligned} \Gamma_{ab}^y &= \delta_a^t [\rho_{Tb}^{\Omega}]_0 + \delta_a^y [\rho_{\Omega b}^{\Omega}]_0 \\ \Gamma_{ab}^z &= \delta_a^t [\rho_{Tb}^{\Omega}]_1 - (\partial_a \gamma_1) [\rho_{Tb}^{\Omega}]_0 + \delta_a^y [\rho_{\Omega b}^{\Omega}]_1 + \delta_a^z [\rho_{\Omega b}^{\Omega}]_0 \\ \Gamma_{ab}^t &= \delta_a^t [\rho_{Tb}^T]_0 + \delta_a^y [\rho_{\Omega b}^T]_0 \end{aligned}$$

where we adopt the notation  $[\rho_{\nu b}^{\mu}]_n$  for the coefficient of  $\lambda^n$  in  $\rho_{\nu b}^{\mu}$ . The Christoffel symbols hence can be written

$$\begin{aligned} \Gamma_{ab}^y &= \delta_a^t A_b^0 + \delta_a^y B_b \\ \Gamma_{ab}^z &= \delta_a^t A_b^1 + \delta_a^y \phi_0 A_b^1 + \delta_a^z (B_b - \phi_0 A_b^0) \\ \Gamma_{ab}^t &= \delta_a^t (C_b - \phi_0 A_b^0 - \phi_{-1} A_b^1) + \delta_a^y (-\phi_0 (B_b - C_b) - A_b^1 \phi_0 \phi_{-1} + \delta_b^y \psi_0 + \delta_b^z \psi_{-1}). \end{aligned}$$

This is the torsion  $\Xi$ -connection, a family of connections parametrised by four arbitrary one-forms  $(A^0, A^1, B, C)$  on  $M$  which generically possess torsion arising from the second derivative of  $f$  via the  $\psi_n$  terms in  $\Gamma_{ab}^t$ . For example,

$$\Gamma_{[yz]}^t = \frac{1}{2} (-\phi_0 (B_z - C_z) - A_z^1 \phi_0 \phi_{-1} + \psi_{-1})$$

cannot be set to zero by a global choice of  $(A^0, A^1, B, C)$  provided  $\psi_{-1} \neq 0$  and provided  $\phi_0$  has vanishing points, which is generically the case.

It is precisely the presence of this torsion which allows the above connection to be made compatible with the clock (23) in stage three of the proof. To carry out this stage we impose  $\nabla\theta = 0$  for the torsion  $\Xi$ -connection.

$$\nabla\theta = 0 \quad \Rightarrow \quad \Gamma_{ab}^t = m\partial_b\theta_a - \Gamma_{ab}^y\phi_0$$

which results in the one-form  $C$  being fixed to be

$$C_b = m\partial_b(m^{-1}) + \phi_{-1}A_b^1$$

which simplifies  $\Gamma_{ab}^t$  to

$$\Gamma_{ab}^t = \delta_a^t(m\partial_b(m^{-1}) - \phi_0 A_b^0) + \delta_a^y(\phi_0 m\partial_b(m^{-1}) - \phi_0 B_b + \delta_b^y\psi_0 + \delta_b^z\psi_{-1}).$$

Thus the moduli space comes equipped with a family of compatible connections and clocks with torsion parametrised by three arbitrary one-forms  $(A^0, A^1, B)$  and one non-vanishing function  $m$ .

In stage four the construction is completed with the calculation of a family of Newton-Cartan metrics compatible with the connections and whose kernels are spanned by the clock. The latter condition,  $h(\theta, \cdot) = 0$ , requires

$$h^{at} + h^{ay}\phi_0 = 0 \quad (30)$$

so it is only necessary to calculate the spatial components  $h^{ij}$  of the metric; one can always reconstruct the other components by factors of  $-\phi_0$ . Moving to the compatibility with the connection,

$$\nabla h = 0 \quad \Rightarrow \quad \partial_b h^{ij} + 2h^{ij}(B_b - \phi_0 A_b^0) = 0. \quad (31)$$

The Frobenius theorem (see, for example, [20]) tells us that there exists a unique solution  $h^{ij}(x^b)$  for given initial data  $h^{ij}(x_0^b)$  iff the one-form  $B - \phi_0 A^0$  is closed. Thus we may henceforth consider  $A^0$  to be free and  $B$  to be constrained to be given by

$$B = \phi_0 A^0 + d\mathcal{B} \quad (32)$$

for an arbitrary function  $\mathcal{B}$  on  $M$ . (We assume the first De Rham cohomology of the relevant domain in  $M$  is trivial; otherwise (32) is reduced to a local statement.) Equation (31) then has solutions

$$h^{ij} = \tilde{h}^{ij} \exp \left\{ -2 \int (B_b - \phi_0 A_b^0) dx^b \right\} = \tilde{h}^{ij} \exp \{-2\mathcal{B}\}$$

where  $\tilde{h}^{ij}$  are constants, and for the purposes of constructing a metric we may choose any  $\tilde{h}^{ij} \in \text{GL}(2, \mathbb{C})$ . Recall that the  $h^{at}$  components can then be found from (30). The final form of the connection is then

$$\begin{aligned} \Gamma_{ab}^y &= \delta_a^t A_b^0 + \delta_a^y \phi_0 A_b^0 + \delta_a^y \partial_b \mathcal{B} \\ \Gamma_{ab}^z &= \delta_a^t A_b^1 + \delta_a^y \phi_0 A_b^1 + \delta_a^z \partial_b \mathcal{B} \\ \Gamma_{ab}^t &= (\delta_a^t + \delta_a^y \phi_0) (m\partial_b(m^{-1}) - \phi_0 A_b^0) - \delta_a^y \phi_0 \partial_b \mathcal{B} + \delta_a^y \delta_b^y \psi_0 + \delta_a^y \delta_b^z \psi_{-1}. \end{aligned}$$

This completes the construction of the Newton-Cartan structures; the free data consists of two one-forms  $(A^0, A^1)$ , two functions  $(\mathcal{B}, m)$  (subject to  $m \neq 0$ ), and a choice of  $\tilde{h}^{ij} \in \text{GL}(2, \mathbb{C})$ .  $\square$

Note that, compared to the case of theorem 3.1, we have a larger family of Newton-Cartan structures (depending on more arbitrary degrees of freedom). This is because we had to use the torsion  $\Xi$ -connection rather than the more powerful torsion-free  $\Lambda$ -connection. It would be pleasing to be able to construct an analogue of the  $\Lambda$ -connection which allows torsion; we defer such prospects to future investigations.

### 3.3 Global Vectors

Which of the many non-relativistic conformal symmetry algebras are singled out on  $M$  by the twistor theory in the same way that Penrose's theory singles out the conformal algebra? For the case of four dimensions it was shown in [19] that the answer is the conformal Newton-Cartan algebra,  $\text{cnc}(3)$ . In three dimensions, though, a naive guess of  $\text{cnc}(2)$  turns out to be incorrect.

In this subsection we'll identify the algebra by computing  $\check{H}^0(Z, TZ)$  for the flat model  $Z = \mathcal{O} \oplus \mathcal{O}(1)$  and pulling the elements back to  $TM \rightarrow M$ . The patching for  $TZ \rightarrow Z$  is

$$\hat{\beta}^{\tilde{\alpha}} = \frac{\partial \hat{Z}^{\tilde{\alpha}}}{\partial Z^{\tilde{\beta}}} \beta^{\tilde{\beta}} \quad (33)$$

where

$$Z^{\tilde{\alpha}} = (T, \Omega, \lambda)^T \quad \hat{Z}^{\tilde{\alpha}} = (\hat{T}, \hat{\Omega}, \hat{\lambda})^T$$

and where we write  $\beta$  for a section of  $TZ$ . The components of (33) read

$$\hat{\beta}^T = \beta^T \quad \hat{\beta}^\Omega = \lambda^{-1} \beta^\Omega - \lambda^{-2} \Omega \beta^\lambda \quad \hat{\beta}^\lambda = -\lambda^{-2} \beta^\lambda.$$

The most general global section is then given by

$$\beta^T = h_0$$

$$\beta^\Omega = d_0 + d_1 \lambda + e_0 \Omega + a_2 \Omega \lambda + b_1 \Omega^2$$

$$\beta^\lambda = a_0 + a_1 \lambda + a_2 \lambda^2 + b_0 \Omega + b_1 \Omega \lambda$$

for nine arbitrary holomorphic functions  $(a_0, a_1, a_2, b_0, b_1, d_0, d_1, e_0, h_0)$  of  $T$ .

Now let  $\tilde{X}$  be a vector field on  $PS' \rightarrow M$ , so we have

$$\tilde{X} = \tilde{X}^\Lambda \frac{\partial}{\partial x^\Lambda} = \tilde{X}^\lambda \frac{\partial}{\partial \lambda} + \tilde{X}^a \frac{\partial}{\partial x^a}$$

over  $U \subset \mathbb{P}^1$ . Then  $\beta = \mu_* \tilde{X}$  is a vector field on  $Z$  with components

$$\beta^\mu = \frac{\partial Z^\mu}{\partial x^\Lambda} \tilde{X}^\Lambda$$

so we have

$$\tilde{X}^t = h_0$$

$$\tilde{X}^\lambda = a_0 + a_1 \lambda + a_2 \lambda^2 + b_0 (y + z\lambda) + b_1 (y + z\lambda) \lambda$$

$$\Rightarrow \tilde{X}^y = d_0 + e_0 y + b_1 y^2 - z a_0 - b_0 y z$$

$$\text{and } \tilde{X}^z = d_1 + (e_0 - a_1) z + a_2 y + b_1 y z - b_0 z^2.$$

We can then push  $\tilde{X}$  down to  $M$  to obtain

$$\nu_* \tilde{X} = h_0 \partial_t + (A^i + B_j^i x^j + x^i C_j x^j) \partial_i$$

where

$$A^y = d_0 \quad A^z = d_1$$

$$B_j^i = \begin{pmatrix} e_0 & -a_0 \\ a_2 & e_0 - a_1 \end{pmatrix}$$

$$C_y = b_1 \quad C_z = -b_0.$$

This is a nine-dimensional Lie algebra under the usual bracket.

We can interpret this algebra heuristically as follows. Take the eight-dimensional algebra of projective vector fields on the two-dimensional spatial fibres, add time translations, and then promote the nine-dimensional algebra to an infinite-dimensional one by allowing the nine components to carry arbitrary holomorphic functions on the time axis. We can write

$$\check{H}^0(Z, TZ) = \left\{ \mathfrak{p}_{\text{eight}}(2, \mathbb{C}) \oplus \left\{ \frac{\partial}{\partial T} \right\} \right\} \otimes H(\mathcal{O}_T)$$

where  $\mathfrak{p}(2, \mathbb{C})$  is the algebra of projective vector fields on the (flat) two-dimensional spatial slices, and where  $H(\mathcal{O}_T)$  are the holomorphic functions on the time axis.

### 3.4 The Relativistic Limit

Since the normal bundle  $N_x = \mathcal{O} \oplus \mathcal{O}(1)$  is stable with respect to Kodaira deformations the Newtonian limit cannot be realised as a jumping phenomenon in the same way as is done in [2] for the case of four dimensions. We can, however, carry out the reverse procedure: if we augment the minitwistor space (with  $N_x = \mathcal{O}(2)$ ) with an additional additive  $\mathcal{O}(-1)$  (so that the normal bundle is taken to be  $\mathcal{O}(-1) \oplus \mathcal{O}(2)$ ) then one can consider a relativistic limit of the Newtonian theory.

**Theorem 3.4.** *Let  $Z_\epsilon \rightarrow \mathbb{P}^1$  be a one-parameter family of rank-two vector bundles with patching*

$$\hat{\zeta} = \lambda\zeta + \epsilon Q$$

$$\hat{Q} = \lambda^{-2}Q.$$

- For  $\epsilon = 0$   $Z_0 = \mathcal{O}(-1) \oplus \mathcal{O}(2)$  and the complex moduli space comes equipped with a three-dimensional non-degenerate conformal structure, as in minitwistor theory.
- For  $\epsilon \neq 0$   $Z_\epsilon = \mathcal{O} \oplus \mathcal{O}(1)$  and the complex moduli space is a three-dimensional Newton-Cartan manifold as described in theorem 3.3.

#### Proof

For  $\epsilon = 0$  the patching for  $Z_0$  is exactly that of  $\mathcal{O}(-1) \oplus \mathcal{O}(2)$  and so the twistor lines are

$$\zeta| = 0$$

and

$$Q| = \xi\lambda^2 - 2z\lambda - \tilde{\xi}$$

which give rise to

$$[g] = dz^2 + d\xi d\tilde{\xi}$$

as expected from minitwistor theory.

For  $\epsilon \neq 0$  we can effect the biholomorphisms

$$\hat{T} = \epsilon^{-1} \hat{\zeta} \quad \hat{\Omega} = -\hat{\lambda}^2 \epsilon^{-1} \hat{\zeta} + \hat{Q}$$

and

$$T = \lambda \epsilon^{-1} \zeta + Q \quad \Omega = -\epsilon^{-1} \zeta$$

which bring the patching to the familiar

$$\hat{T} = T \quad \hat{\Omega} = \lambda^{-1} \Omega$$

and revealing that  $Z_\epsilon = \mathcal{O} \oplus \mathcal{O}(1)$  and placing us in the arena of theorem 3.1.

□

Note that the limit parameter  $\epsilon$  appears only in the complex structures of the one-parameter family of twistor spaces, not in the explicit induced geometry. Thus it does not make sense to consider the limit on the spacetime side of the correspondence in the same way as in four dimensions [2].

### 3.5 On Jumping Hypersurfaces of Gibbons-Hawking Manifolds

We end this section with a tangential result, in which three-dimensional torsional Newton-Cartan manifolds arise on certain hypersurfaces of Gibbons-Hawking manifolds. Recall that a Penrose twistor space is in the Gibbons-Hawking [13] class if it admits a fibration over  $\mathcal{O}(2)$ .

**Theorem 3.5.** *Let  $Z \rightarrow \mathbb{P}^1$  be a twistor space in the Gibbons-Hawking class and let  $(M, g)$  be its associated moduli space with*

$$g = V^{-1} (dt + A)^2 + V (dz^2 + d\xi d\tilde{\xi})$$

*where the Gibbons-Hawking potential  $V$  satisfies  $dV = \star^3 dA$ . On twistor lines satisfying  $V = 0$  the normal bundle is  $\mathcal{O} \oplus \mathcal{O}(2)$  and the twistor-induced local geometry is that of a (generically-torsional)  $(2+1)$ -dimensional Newton-Cartan spacetime, provided that the restriction to  $V = 0$  of the flat three-metric  $dz^2 + d\xi d\tilde{\xi}$  is of rank two.*

#### Proof

Consider first the isomorphism class of the normal bundle to twistor lines. The patching for the normal bundle is

$$\mathcal{F} = \begin{pmatrix} 1 & \frac{\partial f}{\partial Q} \\ 0 & \lambda^{-2} \end{pmatrix}$$

and we can make an expansion

$$\frac{\partial f}{\partial Q} = \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n.$$

(The intersection  $U \cap \hat{U} \subset \mathbb{P}^1$  is an annulus.) The splitting problem is

$$\begin{pmatrix} 1 & \frac{\partial f}{\partial Q} \\ 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 \\ \hat{h}_3 & \hat{h}_4 \end{pmatrix} \begin{pmatrix} \lambda^{m-2} & 0 \\ 0 & \lambda^{-m} \end{pmatrix} \quad (34)$$

and if there exists a holomorphic solution to this for some  $m$  on some line  $X_x$  then the normal bundle to that line is  $\mathcal{O}(2-m) \oplus \mathcal{O}(m)$ . We're interested in the exact form of  $H$  on the set of twistor lines described by  $V = 0$ , which is when  $\gamma_{-1} = 0$  and where we can solve the splitting problem for  $m = 2$ . Henceforth assume that everything is restricted to  $\gamma_{-1} = 0$ .

$$\hat{h}_1 = h_1 + \frac{\partial f}{\partial Q}|_{h_3}$$

$$\hat{h}_2 = \lambda^2 h_2 + \lambda^2 \frac{\partial f}{\partial Q}|_{h_4}$$

$$\hat{h}_3 = \lambda^{-2} h_3$$

$$\hat{h}_4 = h_4.$$

So

$$h_4 = b_0 \quad h_3 = a_0 + a_1 \lambda + a_2 \lambda^2$$

for four functions  $(a_0, a_1, a_2, b_0)$  on  $M$  and

$$h_2 = -b_0 \sum_{n=0}^{\infty} \gamma_n \lambda^n \quad h_1 = c_0 - a_0 \sum_{n=1}^{\infty} \gamma_n \lambda^n - a_1 \sum_{n=0}^{\infty} \gamma_n \lambda^{n+1} - a_2 \sum_{n=0}^{\infty} \gamma_n \lambda^{n+2}.$$

Then

$$\det H = \det \hat{H} = b_0 c_0.$$

One can now set

$$a_0 = a_1 = a_2 = 0$$

and

$$b_0 = c_0 = 1$$

to get a simple solution to (34). The twistor functions are

$$Q| = \xi \lambda^2 - 2z\lambda - \tilde{\xi} \quad \text{for } x^i = (\xi, \tilde{\xi}, z) \in \mathbb{C}^3$$

and

$$T| = t - h(x^i, \lambda)$$

where

$$f| = h - \hat{h}$$

for  $h$  and  $\hat{h}$  holomorphic on  $U$  and  $\hat{U}$  respectively.

Now let  $w = \begin{pmatrix} T \\ Q \end{pmatrix}$  and  $\hat{w} = \begin{pmatrix} \hat{T} \\ \hat{Q} \end{pmatrix}$ ;

$$\begin{aligned} d\hat{w}| &= \mathcal{F}dw| \\ \Rightarrow \quad \hat{H}^{-1}d\hat{w}| &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} H^{-1}dw| \end{aligned}$$

is a global section of  $N \otimes \Lambda^1(M)$  which determines the frame  $(\theta, e^{A'B'})$  for the moduli space.

$$H^{-1}dw| = \begin{pmatrix} 1 & \sum_{n=0}^{\infty} \gamma_n \lambda^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dt - dh \\ d\xi \lambda^2 - 2dz\lambda - d\tilde{\xi} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \theta \\ e^{0'0'}\lambda^2 - 2e^{0'1'}\lambda - e^{1'1'} \end{pmatrix} = \begin{pmatrix} dt - dh + (\sum_{n=0}^{\infty} \gamma_n \lambda^n) [d\xi \lambda^2 - 2dz\lambda - d\tilde{\xi}] \\ d\xi \lambda^2 - 2dz\lambda - d\tilde{\xi} \end{pmatrix}$$

Now consider

$$df| = dh - d\hat{h} = dQ| \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n.$$

$$\Rightarrow dh = \left[ d\xi \sum_{n=0}^{\infty} \gamma_n \lambda^{n+2} - 2dz \sum_{n=0}^{\infty} \gamma_n \lambda^{n+1} - d\tilde{\xi} \sum_{n=1}^{\infty} \gamma_n \lambda^n \right] + \alpha [\gamma_{-2} d\xi + \gamma_0 d\tilde{\xi}]$$

for any choice of  $\alpha$  (parametrising how we choose to share this term between  $dh$  and  $d\hat{h}$ ).

The spatial part of the frame defines a conformal structure in the two remaining spatial dimensions provided that the rank of the restriction of  $dz^2 + d\xi d\tilde{\xi}$  to  $V = 0$  is of rank two, as required in the theorem.

We can now extract the clock:

$$\theta = dt - \alpha \gamma_{-2} d\xi - (1 - \alpha) \gamma_0 d\tilde{\xi} \quad (35)$$

and the triad is the standard flat triad  $(d\xi, dz, d\hat{\xi})$  restricted to  $V = 0$ . Choose  $\alpha = \frac{1}{2}$  for definiteness. The torsion of the Newton-Cartan connection is determined by

$$d\theta = -\frac{1}{2} d\gamma_{-2} \wedge d\xi - \frac{1}{2} d\gamma_0 \wedge d\hat{\xi}$$

which does not generically vanish. □

Note that the torsion originates from  $A$ .

## 4 Some Novel Features in Four Dimensions

### 4.1 The $\Xi$ -Connection for $Z = \mathcal{O} \oplus \mathcal{O}(2)$

The construction in this simple case amounts to taking a global section

$$\chi_{\nu a}^\mu \in \check{H}^0(F|_x, N_x \otimes N_x^* \otimes \Lambda_x^1(M))$$

per point  $x \in M$  and extracting the connection  $\Gamma_{bc}^a$  from

$$\Gamma_{bc}^a \partial_a w^\mu | = \partial_b \partial_c w^\mu | + \chi_{\nu b}^\mu \partial_c w^\nu | + \chi_{\nu c}^\mu \partial_b w^\nu |.$$

For  $Z = \mathcal{O} \oplus \mathcal{O}(2)$  we have

$$N_x \otimes N_x^* = \begin{pmatrix} \mathcal{O} & \mathcal{O}(-2) \\ \mathcal{O}(2) & \mathcal{O} \end{pmatrix}$$

so the most general  $\chi_{\nu a}^\mu$  is

$$\begin{aligned} \chi_{Ta}^T &= A_a & \chi_{Qa}^T &= 0 & \chi_{Qa}^Q &= E_a \\ \chi_{Ta}^Q &= B_a + \lambda C_a + \lambda^2 D_a \end{aligned}$$

for five arbitrary one-forms  $(A_a, B_a, C_a, D_a, E_a)$  on  $M$ . One can then read off the connection components;

$$\begin{aligned} \Gamma_{tt}^t &= 2A_t & \Gamma_{it}^t &= A_i & \Gamma_{ij}^t &= 0 \\ \Gamma_{tt}^\xi &= 2D_t & \Gamma_{tt}^z &= -C_t & \Gamma_{tt}^{\tilde{\xi}} &= -2B_t \\ \Gamma_{jt}^\xi &= D_j + E_t \delta_j^\xi & \Gamma_{jt}^z &= -\frac{1}{2}C_j + E_t \delta_j^z & \Gamma_{jt}^{\tilde{\xi}} &= -B_j + E_t \delta_j^{\tilde{\xi}} \\ \Gamma_{jk}^i &= E_j \delta_k^i + E_k \delta_j^i. \end{aligned}$$

The connection can therefore be any connection provided that  $\Gamma_{ij}^t = 0$  and that the spatial sector is that of a flat projective structure in three dimensions. Note that this includes all generalised Coriolis forces.

### 4.2 Jumps in Four-Dimensional Newtonian Twistor Theory

Jumping phenomena were studied in [17] but are not constrained to the twistor theory of (complexified) Riemannian spacetimes; they can also occur in Newtonian twistor theory.

**Theorem 4.1.** *Let  $Z$  be the total space of an affine line bundle fibred over  $\mathcal{O}(3)$  with patching*

$$\begin{aligned} \hat{\zeta} &= \lambda \zeta + f(S) \\ \hat{S} &= \lambda^{-3} S \end{aligned}$$

where  $f(S)$  is a polynomial of at least quadratic order.  $Z$  is a Newtonian twistor space: the Kodaira moduli space  $M$  of global sections is locally a complex Newton-Cartan spacetime.

- The normal bundle is generically  $N_x = \mathcal{O} \oplus \mathcal{O}(2)$ .
- At special points the normal bundle jumps to  $\mathcal{O}(-1) \oplus \mathcal{O}(3)$ ; at these jumping points the time axis in the moduli space becomes singular.

The special points are characterised by the vanishing of the three two-forms induced on  $M$  by the global two form  $d\hat{\zeta} \wedge d\hat{S}$ , or equivalently by the vanishing of the constant term  $\gamma_0$  in the Laurent expansion of  $\frac{\partial f}{\partial S}|$ .



## Proof

Write the global sections of  $\mathcal{O}(3)$  as

$$S| = x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3.$$

First we will consider the normal bundle; we will show that generically the isomorphism class of  $N_x$  is  $\mathcal{O} \oplus \mathcal{O}(2)$ . The patching for  $N_x$  is

$$\mathcal{F} = \begin{pmatrix} \lambda & \frac{\partial f}{\partial S}| \\ 0 & \lambda^{-3} \end{pmatrix}$$

and the splitting problem to be solved is

$$\hat{h}_1 = \lambda h_1 + \frac{\partial f}{\partial S}|h_3 \quad (36)$$

$$\hat{h}_2 = \lambda^3 h_2 + \lambda^2 \frac{\partial f}{\partial S}|h_4 \quad (37)$$

$$\hat{h}_3 = \lambda^{-3} h_3$$

$$\hat{h}_4 = \lambda^{-1} h_4.$$

As usual we put

$$h_3 = \sum_{n=0}^3 a_n \lambda^n \quad \text{and} \quad h_4 = b_0 + b_1 \lambda.$$

To proceed further we make an expansion:

$$\frac{\partial f}{\partial S}| = \gamma_0 + \sum_{n=1}^{\infty} \gamma_n \lambda^n$$

where  $\gamma_0$  is a function of  $x_0$  only. For a global solution of (37) we then require

$$b_0 \gamma_0 = 0, \quad (38)$$

leaving

$$\hat{h}_2 = 0.$$

We also have

$$\hat{h}_1 = \gamma_0 a_0.$$

The determinant is then

$$\det \hat{H} = \gamma_0 a_0 b_1,$$

and we conclude that  $N_x = \mathcal{O} \oplus \mathcal{O}(2)$  for  $\gamma_0 \neq 0$ . A straightforward calculation then shows that  $N_x = \mathcal{O}(-1) \oplus \mathcal{O}(3)$  for any twistor line  $X_x$  with  $\gamma_0 = 0$ .

Since we generically have  $N_x = \mathcal{O} \oplus \mathcal{O}(2)$  we anticipate that the moduli space will be a Newton-Cartan spacetime; to see this in detail we must construct the twistor functions. In this case it is very straightforward: over  $\hat{U}$  we have

$$\hat{\zeta} = f(S|(\lambda = 0)) := \mathcal{T}(x_0)$$

and

$$\hat{S}| = x_0 \hat{\lambda}^3 + x_1 \hat{\lambda}^2 + x_2 \hat{\lambda} + x_3.$$

We note that  $\gamma_0 = \frac{dT}{dx_0}$ . The geometry induced on the moduli space can be found by identifying null vectors as those tangent to alpha surfaces. The clock is therefore

$$\theta = \alpha(x_0) \gamma_0(x_0) dx_0$$

for any non-vanishing  $\alpha$  and the conformal covariant Galilean metric is

$$h^{-1} = \beta(dx_2^2 - 4dx_1 dx_3)$$

for any non-vanishing  $\beta$ . The geometry is therefore Newton-Cartan at generic points. At the jumping points with  $\gamma_0 = 0$ , though, the clock vanishes.

One can construct a map

$$x_0 \mapsto t(x_0) = \mathcal{T}(x_0) \tag{39}$$

taking  $M$  to a more usual non-jumping Newton-Cartan spacetime but the map is not a diffeomorphism; it eliminates the jumping points. This situation is entirely analogous to the map taking the jumping spacetimes from [17] to Gibbons-Hawking form.

The three two-forms arising from the restriction to twistor lines of

$$d\hat{\zeta} \wedge d\hat{S} = \lambda^{-2} d\zeta \wedge dS$$

clearly vanish on and only on the jumping points. □

### Example

The simplest example of a jumping Newtonian twistor space is when  $f(S) = \frac{1}{2}S^2$ . The conformal clock admits a representative

$$\theta = x_0 dx_0 ,$$

vanishing at one point  $x_0 = 0$ . The map (39) is

$$t = \frac{1}{2}x_0^2 ,$$

and so the Newton-Cartan spacetime is thus a 2-fold cover of the standard Newton-Cartan spacetime (with time coordinate  $t$ ), branched over the spatial fibre  $t = 0$ .

## 5 Five Dimensions

In this section we'll study some five-dimensional Kodaira families. Mostly we'll be concerned with the Newtonian theory, for which the normal bundle will be  $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ , though later in this section we will also consider the normal bundle  $\mathcal{O}(4)$ , for which the moduli space comes equipped with relativistic geometry in accordance with theorem 2.1.

### 5.1 Galilean Structures and Canonical Connections

We'll begin by constructing the geometry on the moduli space for the undeformed Newtonian case, equipped with its canonical  $\Lambda$ -connection.

**Theorem 5.1.** *Let  $Z = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ . The induced geometry on the moduli space of global sections of  $Z \rightarrow \mathbb{P}^1$  is that of a complex five-dimensional Newton-Cartan spacetime  $(M, h, \theta, \nabla)$ , where the connection components of  $\nabla$  depend (up to diffeomorphisms of the time axis) on one conformal factor and seven arbitrary functions,*

- *four of which are the Newtonian gravitational force  $\Gamma_{tt}^i$ ;*
- *and the remaining three of form an anti-self-dual spatial two-form  $W_{ij}$  describing Coriolis forces.*

#### Proof

The normal bundle to twistor lines is  $N_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ , which satisfies  $\check{H}^1(\mathbb{P}^1, N_x) = 0$ . By the Kodaira theorem [5] the moduli space  $M$  is therefore a complex manifold, with dimension  $\dim \check{H}^0(\mathbb{P}^1, N_x) = 5$ .

To construct the conformal Galilean structure  $(h, \theta)$  we first find the twistor lines explicitly. In homogeneous coordinates the patching is

$$\hat{T} = T \quad (\text{weight zero})$$

and

$$\hat{\omega}^A = \omega^A \quad (\text{weight one}),$$

with twistor lines

$$T| = t \quad \text{and} \quad \omega^A| = x^{AA'} \pi_{A'}$$

for coordinates  $(t, x^{AA'})$  on  $M$ . To find the frame we need to solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 & \hat{h}_3 \\ \hat{h}_4 & \hat{h}_5 & \hat{h}_6 \\ \hat{h}_7 & \hat{h}_8 & \hat{h}_9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

for the most general  $H$  and  $\hat{H}$ . The solution is

$$H^{-1} = \begin{pmatrix} \frac{1}{mk} \sum_{p=0}^1 \frac{1}{m} (k_2 b_p - k_4 a_p) \lambda^p & 0 & 0 \\ \frac{1}{mk} \sum_{p=0}^1 (k_3 a_p - k_1 b_p) \lambda^p & \frac{k_4}{k} & -\frac{k_2}{k} \\ & -\frac{k_3}{k} & \frac{k_1}{k} \end{pmatrix}.$$

for nine arbitrary functions  $(m, a_0, a_1, b_0, b_1, k_1, k_2, k_3, k_4)$  on  $M$  which must be chosen such that  $m \neq 0$  and  $k := (k_1 k_4 - k_2 k_3) \neq 0$  anywhere. These functions will determine the ambiguity in the moduli space geometry. The frame section is therefore given by

$$v = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ \frac{1}{mk} \sum_{p=0}^1 (k_2 b_p - k_4 a_p) \lambda^p & \frac{k_4}{k} & -\frac{k_2}{k} \\ \frac{1}{mk} \sum_{p=0}^1 (k_3 a_p - k_1 b_p) \lambda^p & -\frac{k_3}{k} & \frac{k_1}{k} \end{pmatrix} \begin{pmatrix} dt \\ dx^{01'} + dx^{00'} \lambda \\ dx^{11'} + dx^{10'} \lambda \end{pmatrix}$$

from which we can read-off the frame

$$\begin{aligned} \theta &= m^{-1} dt \\ e^{00'} &= \frac{k_4}{k} dx^{00'} - \frac{k_2}{k} dx^{10'} + \frac{1}{mk} (k_2 b_1 - k_4 a_1) dt \\ e^{01'} &= \frac{k_4}{k} dx^{01'} - \frac{k_2}{k} dx^{11'} + \frac{1}{mk} (k_2 b_0 - k_4 a_0) dt \\ e^{10'} &= -\frac{k_3}{k} dx^{00'} + \frac{k_1}{k} dx^{10'} + \frac{1}{mk} (k_3 a_1 - k_1 b_1) dt \\ e^{11'} &= -\frac{k_3}{k} dx^{01'} + \frac{k_1}{k} dx^{11'} + \frac{1}{mk} (k_3 a_0 - k_1 b_0) dt, \end{aligned}$$

leading to the natural decomposition of the tangent bundle

$$TM = \mathbb{C} \oplus (\mathbb{S} \otimes \mathbb{S}').$$

The conformal clock is therefore  $\theta = m^{-1} dt$ , and the conformal *covariant* metric is

$$h^{-1} = \epsilon_{AB} \epsilon_{A'B'} e^{AA'} \otimes e^{BB'}$$

$$\begin{aligned} \Rightarrow \quad h^{-1} &= k^{-1} \left( dx^{00'} dx^{11'} - dx^{10'} dx^{01'} \right) \\ &\quad + m^{-1} k^{-1} dt \left( -b_0 dx^{00'} + b_1 dx^{01'} + a_0 dx^{10'} - a_1 dx^{11'} \right) \\ &\quad + m^{-2} k^{-1} (a_1 b_0 - a_0 b_1) dt^2 \end{aligned}$$

The metric is rank-four, as one would expect for a five-dimensional Newton-Cartan spacetime. The contravariant metric is obtained as the projective inverse in the following way. First find a vector  $U$  such that

$$\theta(U) = 1 \quad \text{and} \quad h^{-1}(U, \cdot) = 0,$$

which uniquely determines

$$U = m \partial_t + b_0 \partial_{11'} + b_1 \partial_{10'} + a_0 \partial_{01'} + a_1 \partial_{00'}.$$

The contravariant metric  $h$  is then the unique solution to

$$h^{ab} h_{bc} = \delta_c^a - U^a \theta_c \quad \text{and} \quad h(\theta, \cdot) = 0,$$

which determines

$$h = k^{-1} \epsilon^{AB} \epsilon^{A'B'} \frac{\partial}{\partial x^{AA'}} \otimes \frac{\partial}{\partial x^{BB'}}.$$

Thus we have a Galilean structure  $(M, h, \theta)$  depending on arbitrary functions. (We could also equivalently have used the more traditional twistor theory method of calculating the null vectors from the twistor functions as was done for the case of four dimensions in [2].)

It only remains to calculate the physical induced connection, i.e. we must construct the  $\Lambda$ -connection. Denote by

$$\hat{w}^\mu = \begin{pmatrix} \hat{T} \\ \hat{\omega}^A/\pi_{0'} \end{pmatrix} \quad \text{and} \quad w^\mu = \begin{pmatrix} T \\ \omega^A/\pi_{1'} \end{pmatrix} \quad (40)$$

column vectors of inhomogeneous twistor coordinates on the fibres, and for ease of notation set

$$x^{AA'} = x^i = \begin{pmatrix} v & u \\ y & x \end{pmatrix}. \quad (41)$$

Following the discussion in section 1.2 the construction of  $\nabla$  occurs in two stages, the first being the solution of the splitting problem

$$\mathcal{F}_{\nu\rho}^\mu = -\hat{\sigma}_{\alpha\beta}^\mu \mathcal{F}_\nu^\alpha \mathcal{F}_\rho^\beta + \mathcal{F}_\gamma^\mu \sigma_{\nu\rho}^\gamma \quad (42)$$

for a 0-cochain  $\{\sigma\}$  of  $N \otimes (N^* \odot N^*) \rightarrow \mathbb{P}^1$ , where

$$\mathcal{F}_\nu^\alpha = \frac{\partial \hat{w}^\alpha}{\partial w^\nu} \Big| \quad \text{and} \quad \mathcal{F}_{\nu\rho}^\mu = \frac{\partial^2 \hat{w}^\mu}{\partial w^\nu \partial w^\rho} \Big|.$$

The solution of (42) depends on nine arbitrary functions (on  $M$ ) because

$$\check{H}^0(\mathbb{P}^1, N \otimes (N^* \odot N^*)) = \mathbb{C}^9,$$

and is explicitly given by

$$\begin{aligned} \hat{\sigma}_{AB}^T = \sigma_{AB}^T = 0 \quad \hat{\sigma}_{AT}^T = \sigma_{AT}^T = 0 \quad \hat{\sigma}_{BC}^A = \sigma_{BC}^A = 0 \\ \hat{\sigma}_{TT}^T = \sigma_{TT}^T = \Sigma \quad \hat{\sigma}_{BT}^A = \sigma_{BT}^A = \chi_B^A \\ \hat{\sigma}_{TT}^A = \lambda^{-1} \phi^A + \psi^A \quad \sigma_{TT}^A = \phi^A + \lambda \psi^A. \end{aligned}$$

(The nine functions are  $\Sigma, \phi^A, \psi^A$ , and  $\chi_B^A$ .)

The second stage of the construction is the reading-off of  $\Gamma_{bc}^a$  from the map  $T^{[2]}M \rightarrow TM$  determined by  $\{\sigma\}$  via the Kodaira isomorphism  $TM = \check{H}^0(\mathbb{P}^1, N)$ . Concretely we read off  $\Gamma_{bc}^a(x^d)$  from

$$\Gamma_{bc}^a \partial_a w^\mu \Big| = \partial_b \partial_c w^\mu \Big| + \sigma_{\nu\rho}^\mu \partial_b w^\nu \Big| \partial_c w^\rho \Big|,$$

giving us

$$\Gamma_{tt}^t = \Sigma \quad (43)$$

$$\begin{aligned} \Gamma_{tt}^u = \phi^0 \quad \Gamma_{tt}^v = \psi^0 \quad \Gamma_{tt}^x = \phi^1 \quad \Gamma_{tt}^y = \psi^1 \\ \Gamma_{ut}^u = \Gamma_{vt}^v = \chi_0^0 \quad \Gamma_{xt}^x = \Gamma_{yt}^y = \chi_1^1 \\ \Gamma_{xt}^u = \Gamma_{yt}^v = \chi_1^0 \quad \Gamma_{ut}^x = \Gamma_{vt}^y = \chi_0^1 \end{aligned} \quad (44)$$

with all other components of  $\Gamma_{bc}^a$  vanishing. Two of these functions are related to ones we already have by the compatibility conditions

$$\nabla \theta = 0 \quad \nabla h = 0,$$

which give us

$$\Sigma = -\partial_t \ln m \quad \text{and} \quad \text{tr}(\chi) = \chi_0^0 + \chi_1^1 = -\frac{1}{2} \partial_t \ln k ,$$

as well as

$$\frac{\partial m}{\partial x^{AA'}} = \frac{\partial k}{\partial x^{AA'}} = 0$$

so these two factors are functions of time only. The function  $m$  can be set to one without loss of generality by a diffeomorphism of the time axis.

The four components  $\Gamma_{tt}^i$  are completely arbitrary (given in terms of  $\phi^A$  and  $\psi^A$ ), whilst the remaining  $\Gamma_{jt}^i$  components depend on three arbitrary functions from the traceless part of  $\chi_B^A$ . Thus we get only three functions' worth of  $\Gamma_{jt}^i$  instead of the most general case depending on six functions. Given that this is twistor theory it is perhaps no surprise that the three functions form an anti-self-dual two-form on spatial fibres. Concretely we have

$$\Gamma_{jt}^i = \delta^{ik} W_{jk}$$

with

$$W = (\chi_0^0 - \chi_1^1) [du \wedge dy + dx \wedge dv] + 2\chi_1^0 [dx \wedge dy] + 2\chi_0^1 [dv \wedge du] . \quad (45)$$

It is then straightforward to check that (45) is the most general anti-self-dual two-form on spatial fibres with respect to a volume-form

$$\epsilon_{\text{space}} = dx \wedge dy \wedge dv \wedge du ,$$

completing the proof. □

We thus conclude that Newtonian twistor theory in five dimensions admits generalised Coriolis connection components as arbitrary functions in its  $\Lambda$ -connection, but only *half* of them.

### $\Xi$ -Connection

As described in section 1.2 the calculation amounts to taking a global section  $\chi_{\nu a}^\mu$  of  $N_x \otimes N_x^* \otimes \Lambda_x^1(M)$  per point  $x \in M$ . For  $N_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$  we have

$$N_x \otimes N_x^* = \begin{pmatrix} \mathcal{O} & \mathcal{O}(-1) & \mathcal{O}(-1) \\ \mathcal{O}(1) & \mathcal{O} & \mathcal{O} \\ \mathcal{O}(1) & \mathcal{O} & \mathcal{O} \end{pmatrix}$$

giving us

$$\begin{aligned} \chi_{Ta}^T &= C_a & \chi_{Aa}^T &= 0 & \chi_{Ta}^A &= A_a^{AA'} \pi_{A'} \\ \chi_{Ba}^A &= B_{Ba}^A \end{aligned}$$

for nine arbitrary one-forms  $(C, A^{AA'}, B_B^A)$  on  $M$  constituting forty-five arbitrary functions. We extract the connection symbols by reading off from

$$\Gamma_{bc}^a \partial_a w^\mu = \partial_b \partial_c w^\mu + \chi_{\nu b}^\mu \partial_c w^\nu + \chi_{\nu c}^\mu \partial_b w^\nu ,$$

which gives us

$$\begin{aligned}\Gamma_{bc}^t &= \delta_b^t C_c + \delta_c^t C_b \\ \Gamma_{tt}^{AA'} \pi_{A'} &= 2A_t^{AB'} \pi_{B'} \\ \Gamma_{BB't}^{AA'} \pi_{A'} &= A_{BB'}^{AC'} \pi_{C'} + B_{Bt}^A \pi_{B'} \\ \Gamma_{BB'CC'}^{AA'} \pi_{A'} &= B_{CBB'}^A \pi_{C'} + B_{BCC'}^A \pi_{B'}.\end{aligned}$$

The  $\Xi$ -connection therefore contains every connection which has  $\Gamma_{ij}^t = 0$  and for which the four-dimensional spatial sector resembles that of the  $\Xi$ -connection (6) for  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  as discussed in 1.2.

## 5.2 Deformations in the Newtonian Theory

In this subsection we'll study deformations of the form

$$\hat{T} = T + \epsilon f(\Omega^A, \lambda) \quad (46)$$

over the total space of  $\mathcal{O}(1) \oplus \mathcal{O}(1)$ , where  $\epsilon$  is a deformation parameter. Of course, from one point of view this is nothing more than a Ward bundle on the twistor space for flat four-dimensional spacetime. The approach adopted in this paper is instead to study the geometry of the full five-dimensional moduli space of global sections.

Given that the normal bundle  $N_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$  is stable with respect to all Kodaira deformations we must therefore possess a five-dimensional Galilean structure, but the connection is more subtle.  $\check{H}^1(\mathbb{P}^1, N_x \otimes (N_x^* \odot N_x^*)) \neq 0$ , so some deformations will result in a moduli space which does *not* possess a  $\Lambda$ -connection. Like in the three-dimensional case, what is going wrong is that a  $\Lambda$ -connection is, by construction, torsion-free, and deformations of the form (46) give rise to moduli spaces whose Newton-Cartan structures possess torsion.

**Theorem 5.2.** *Let  $Z$  be a complex four-fold fibred over  $\mathbb{P}^1$  with patching given by (46) whose five-parameter family of global sections  $X_x$  have normal bundle  $N_x = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ . The moduli space  $M$  of those sections is a complex five-dimensional manifold equipped with a Galilean structure with torsion whose clock admits a representative with*

$$d\theta = \epsilon \left\{ \frac{1}{2\pi i} \oint \frac{\partial^2 f}{\partial \omega^A \partial \omega^B} \Big|_{\frac{\pi_{B'}}{\pi_0}} \pi \cdot d\pi \right\} dx^{BB'} \wedge dx^{A1'}.$$

(Recall that  $\omega^A$  are the homogeneous versions of  $\Omega^A$ .)

### Proof

Take the global sections of the base  $\mathcal{O}(1) \oplus \mathcal{O}(1)$  to be  $x^{AA'} \pi_{A'}$  as in (41), or in inhomogeneous coordinates

$$\Omega^0| = u + v\lambda \quad \Omega^1| = x + y\lambda.$$

Now restrict  $f$  to these lines and expand it in a Laurent series in  $\lambda$ ;

$$f| = \sum_{n=-\infty}^{\infty} \gamma_n \lambda^n \quad \text{for} \quad \gamma_n = \frac{1}{2\pi i} \oint f(\Omega^A|, \lambda) \lambda^{-(1+n)} d\lambda.$$

The global sections of  $Z \rightarrow \mathbb{P}^1$  are then completed by

$$T| = t - \epsilon \sum_{n=1}^{\infty} \gamma_n \lambda^n.$$

The next task is to calculate the frame section. We must solve

$$\begin{pmatrix} 1 & \epsilon \frac{\partial f}{\partial \Omega^0} | & \epsilon \frac{\partial f}{\partial \Omega^1} | \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 & \hat{h}_2 & \hat{h}_3 \\ \hat{h}_4 & \hat{h}_5 & \hat{h}_6 \\ \hat{h}_7 & \hat{h}_8 & \hat{h}_9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Written out in full we must therefore solve

$$\begin{aligned} \hat{h}_1 &= h_1 + \epsilon \frac{\partial f}{\partial \Omega^0} | h_4 + \epsilon \frac{\partial f}{\partial \Omega^1} | h_7 \\ \hat{h}_2 &= \lambda h_2 + \epsilon \lambda \frac{\partial f}{\partial \Omega^0} | h_5 + \epsilon \lambda \frac{\partial f}{\partial \Omega^1} | h_8 \\ \hat{h}_3 &= \lambda h_3 + \epsilon \lambda \frac{\partial f}{\partial \Omega^0} | h_6 + \epsilon \lambda \frac{\partial f}{\partial \Omega^1} | h_9 \\ \hat{h}_4 &= \lambda^{-1} h_4 & \hat{h}_7 &= \lambda^{-1} h_7 \\ \hat{h}_5 &= h_5 & \hat{h}_6 &= h_6 & \hat{h}_8 &= h_8 & \hat{h}_9 &= h_9. \end{aligned}$$

Put

$$\begin{pmatrix} h_5 & h_6 \\ h_8 & h_9 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$$

and

$$\begin{pmatrix} h_4 \\ h_7 \end{pmatrix} = \begin{pmatrix} a_0 + a_1 \lambda \\ b_0 + b_1 \lambda \end{pmatrix}.$$

Now expand the derivatives of  $f$  (restricted to twistor lines) in Laurent series;

$$\frac{\partial f}{\partial \Omega^A} | = \sum_{n=-\infty}^{\infty} \phi_{n,A} \lambda^n \quad \text{for} \quad \phi_{n,A} = \frac{1}{2\pi i} \oint \frac{\partial f}{\partial \Omega^A} | \lambda^{-(1+n)} d\lambda.$$

We then (uniquely) obtain

$$\begin{aligned} h_2 &= -\epsilon \sum_{n=0}^{\infty} (\phi_{n,0} k_1 + \phi_{n,1} k_3) \lambda^n \\ h_3 &= -\epsilon \sum_{n=0}^{\infty} (\phi_{n,0} k_2 + \phi_{n,1} k_4) \lambda^n \end{aligned}$$

and we can solve for the remaining piece  $h_1$  up to the arbitrary function  $m$  to obtain

$$h_1 = m - \epsilon \sum_{n=0}^{\infty} \lambda^n [\phi_{n,0} (a_0 + a_1 \lambda) + \phi_{n,1} (b_0 + b_1 \lambda)].$$

Define  $k = k_1 k_4 - k_2 k_3$ . The determinant of  $H$  is then given by

$$\det H = mk$$



so we must impose  $m \neq 0$  and  $k \neq 0$ .

We can then calculate

$$\begin{aligned}
(H^{-1})_T^T &= m^{-1} \\
(H^{-1})_A^T &= \epsilon m^{-1} \sum_{n=0}^{\infty} \phi_{n,A} \lambda^n \\
(H^{-1})_T^A &= \left( \frac{1}{mk} \sum_{p=0}^1 (k_2 b_p - k_4 a_p) \lambda^p \right) \\
(H^{-1})_B^A &= \begin{pmatrix} \frac{k_4}{k} + \frac{\epsilon}{mk} \sum_{p=0}^1 \sum_{n=0}^{\infty} \phi_{n,0} (k_2 b_p - k_4 a_p) \lambda^{n+p} & -\frac{k_2}{k} + \frac{\epsilon}{mk} \sum_{p=0}^1 \sum_{n=0}^{\infty} \phi_{n,1} (k_2 b_p - k_4 a_p) \lambda^{n+p} \\ -\frac{k_3}{k} + \frac{\epsilon}{mk} \sum_{p=0}^1 \sum_{n=0}^{\infty} \phi_{n,0} (k_3 a_p - k_1 b_p) \lambda^{n+p} & \frac{k_1}{k} + \frac{\epsilon}{mk} \sum_{p=0}^1 \sum_{n=0}^{\infty} \phi_{n,1} (k_3 a_p - k_1 b_p) \lambda^{n+p} \end{pmatrix}.
\end{aligned}$$

The clock is therefore given by

$$\begin{aligned}
\theta &= m^{-1} dT| + m^{-1} \epsilon \sum_{n=0}^{\infty} \phi_{n,A} \lambda^n d\Omega^A| \\
\Rightarrow \quad \theta &= m^{-1} \left( dt - \epsilon \sum_{n=1}^{\infty} d\gamma_n \lambda^n + \epsilon \sum_{n=0}^{\infty} \phi_{n,A} \lambda^n d\Omega^A| \right).
\end{aligned}$$

Now, we have

$$d\gamma_n = \phi_{n,A} dx^{A1'} + \phi_{n-1,A} dx^{A,0'}$$

so

$$\theta = m^{-1} \left( dt + \epsilon \phi_{0,A} dx^{A1'} \right).$$

As in the five-dimensional case this is not closed for any  $m \neq 0$  (and  $\epsilon \neq 0$ ). Now take a representative with  $m = 1$ ; we then have

$$\begin{aligned}
d\theta &= \epsilon \partial_{BB'} \phi_{0,A} dx^{BB'} \wedge dx^{A1'} \\
\Rightarrow \quad d\theta &= \epsilon \left\{ \frac{1}{2\pi i} \oint \frac{\partial^2 f}{\partial \omega^A \partial \omega^B} \Big|_{\frac{\pi_{B'}}{\pi_{0'}}} \pi \cdot d\pi \right\} dx^{BB'} \wedge dx^{A1'}.
\end{aligned}$$

The Newton-Cartan metric arises, as in theorem 5.1, from the projective inverse of degenerate covariant metric arising from the frame section, completing the construction of a Galilean structure with torsion.

□

In theorem 5.2 we chose to merely construct the torsional Galilean structure, exhibiting the torsion via the non-closure of the clock. We could, however, go further and explicitly construct the torsion  $\Xi$ -connection of section 1.2.3 as was done for the three-dimensional case in theorem 3.3. In the interests of brevity we omit this cumbersome calculation.

### 5.3 Global Vectors

In this section we will study the image on  $M$  of  $\check{H}^0(Z, TZ)$  for  $Z = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ .

Let  $Z^\alpha$  run over  $w^\mu$  and  $\lambda$ , with analogous definitions for  $\hat{U}$ . The patching for  $TZ \rightarrow Z$  is

$$\hat{V}^\alpha = \frac{\partial \hat{Z}^\alpha}{\partial Z^\beta} V^\beta \quad \Rightarrow \quad \begin{aligned} \hat{\beta}^T &= \beta^T \\ \hat{\beta}^A &= \lambda^{-1} \beta^A - \lambda^{-2} w^A \beta^\lambda, \\ \hat{\beta}^\lambda &= -\lambda^{-2} \beta^\lambda \end{aligned} \quad , \quad (47)$$

and there is one global function of weight zero to consider,  $\hat{T} = T$ . The general global section of (47) is

$$\begin{aligned} \beta &= a(T) \frac{\partial}{\partial T} + (h^A(T) + g^A(T)\lambda + j_B^A(T)w^B + d(T)\lambda w^A + f_B(T)w^B w^A) \frac{\partial}{\partial w^A} \\ &\quad + (b(T) + c(T)\lambda + d(T)\lambda^2 + e_A(T)w^A + f_A(T)w^A \lambda) \frac{\partial}{\partial \lambda} \quad , \quad (48) \end{aligned}$$

depending on sixteen (holomorphic) functions of  $T$ . The global vector algebra has the decomposition

$$\check{H}^0(Z, TZ) = \left\{ \begin{array}{c} \mathfrak{sl}(4, \mathbb{C}) \\ \text{fifteen} \end{array} \oplus \left\{ \frac{\partial}{\partial T} \right\} \right\} \otimes H(\mathcal{O}_T)$$

into conformal symmetries of the flat degenerate metric and time translations, where  $H(\mathcal{O}_T)$  are the holomorphic functions on the time axis.

### 5.4 Alpha-Surfaces in the Relativistic Theory

Via theorem 2.1 one can construct some five-dimensional Riemannian manifolds  $M$  as the Kodaira families of curves with normal bundle  $\mathcal{O}(4)$  in a complex two-fold  $Z$ , the flat model being  $Z = \mathcal{O}(4)$ . In this tangential subsection we'll consider how the conformal structure on  $M$  arises via the presence of alpha surfaces. The construction is considerably more complicated than that of the frame advocated throughout this paper, but its existence is nonetheless reassuring.

The (inhomogeneous) patching for  $\mathcal{O}(4)$  is

$$\hat{S} = \lambda^{-4} S$$

and so the global sections are

$$S| = t + 4u\lambda + 6x\lambda^2 + 4v\lambda^3 + w\lambda^4$$

for  $x^a = (t, u, x, v, w) \in M$ .

The direct calculation of the frame (the case  $n = 2$  in theorem 2.1) is trivial in the flat case and clearly gives us  $e^{A'B'C'D'} = dx^{A'B'C'D'}$  and hence every tensor in the span of the frame, but here we'll consider alpha surfaces.

The null vectors  $\delta x^a$  are those for which

$$0 = \delta t + 4\delta u\lambda + 6\delta x\lambda^2 + 4\delta v\lambda^3 + \delta w\lambda^4$$

has a unique solution in  $\lambda$ . The classical theory of quartic equations tells us that this happens when  $\delta x^a$  solves simultaneously the three conditions

$$\begin{aligned}\Delta_6 &= -\delta t^3 \delta w^3 + 12\delta t^2 \delta u \delta v \delta w^2 + 27\delta t^2 \delta v^4 - 54\delta t^2 \delta v^2 \delta w \delta x \\ &\quad + 18\delta t^2 \delta w^2 \delta x^2 + 6\delta t \delta u^2 \delta v^2 \delta w - 54\delta t \delta u^2 \delta w^2 \delta x - 108\delta t \delta u \delta v^3 \delta x \\ &\quad + 180\delta t \delta u \delta v \delta w \delta x^2 + 54\delta t \delta v^2 \delta x^3 - 81\delta t \delta w \delta x^4 + 27\delta u^4 \delta w^2 \\ &\quad + 64\delta u^3 \delta v^3 - 108\delta u^3 \delta v \delta w \delta x - 36\delta u^2 \delta v^2 \delta x^2 + 54\delta u^2 \delta w \delta x^3 = 0 \\ \Delta_4 &= -\delta t \delta w^3 + 4\delta u \delta v \delta w^2 + 12\delta v^4 - 24\delta v^2 \delta w \delta x + 9\delta w^2 \delta x^2 = 0 \\ \Delta_2 &= -\delta t \delta w + 4\delta u \delta v - 3\delta x^2 = 0.\end{aligned}$$

**Claim:** The vanishing of  $\Delta_6$ ,  $\Delta_4$ , and  $\Delta_2$  is equivalent to  $\delta x^a$  falling into the union of the kernels of the span of the frame  $e^{A'B'C'D'}$ .

- The latter condition  $\Delta_2 = 0$  is exactly what one would expect for the metric, giving us the symmetric two-form  $g$  above. (It agrees exactly with the  $g$  one would calculate from the direct frame method.) Concretely,

$$[g] = -\delta t \delta w + 4\delta u \delta v - 3\delta x^2.$$

- The vanishing of  $\Delta_2$  and  $\Delta_6$  simultaneously is equivalent to  $\delta x^a$  lying in the kernel of both  $g$  and a symmetric three-form  $\mathcal{G}_3$  (which is also consistent with the direct frame calculation). In particular

$$[\mathcal{G}_3(\delta x^a, \delta x^a, \delta x^a)]^2 \propto \Delta_6 \quad \text{when } \Delta_2 = 0,$$

for

$$\mathcal{G}_3(\delta x^a, \delta x^a, \delta x^a) = -\delta t \delta v^2 + \delta t \delta w \delta x - \delta u^2 \delta w + 2\delta u \delta v \delta x - \delta x^3.$$

- Finally, when  $\Delta_2 = \Delta_6 = 0$  the vanishing of  $\Delta_4$  is equivalent to having

$$[\delta x \delta w - \delta v^2 = 0 \quad \text{and} \quad \delta t \delta x - \delta u^2 = 0] \quad \text{and either} \quad \delta w \delta t - \delta x^2 = 0 \quad \text{or} \quad \delta u \delta v - \delta x^2 = 0. \quad (49)$$

These (effectively three) conditions are the requirement that  $\delta x^a$  lie in the kernel of three rank-three symmetric two-forms which we identify as  $e^{0'0'}{}_{A'B'} \otimes e^{0'0'}{}_{A'B'}$ ,  $e^{0'1'}{}_{A'B'} \otimes e^{0'1'}{}_{A'B'}$ , and  $e^{1'1'}{}_{A'B'} \otimes e^{1'1'}{}_{A'B'}$ .

- The rest of the canonical symmetric two-forms  $e^{C'D'}{}_{A'B'} \otimes e^{E'F'}{}_{A'B'}$  also arise, but as redundant conditions equivalent to (49). (One could choose to isolate three other conditions, say one rank-three symmetric two-form and two rank-four symmetric two-forms, and fit those to canonical forms instead, but for concreteness we have chosen the three rank-three symmetric two-forms.)

Thus we can obtain the induced metric via either direct calculation or (in a more complicated fashion) by the usual twistor theory arguments. Once one has the frame one has every canonical form discussed above.

## Acknowledgments

I would like to thank Maciej Dunajski for stimulating discussions and I am grateful to STFC and DAMTP for financial support.

## References

- [1] Merkulov S, “Relative deformation theory and differential geometry”, in *Twistor theory*, Marcel Dekker, New York. (1995)
- [2] Dunajski M and Gundry J, “Non-relativistic twistor theory and Newton-Cartan geometry”, *Comm. Math. Phys.* **342** 1043-1074 (2016)
- [3] Hitchin N, “Complex manifolds and Einstein’s equations”, *Twistor geometry and nonlinear systems (Primorsko, 1980)*, 73–99, *Lecture Notes in Math.* 970, Springer, Berlin-New York. (1982)
- [4] Duval C and Horvathy P, “Non-relativistic conformal symmetries and Newton-Cartan structures”, *J.Phys.* **A42**, 465206. (2009)
- [5] Kodaira K, “On stability of compact submanifolds of complex manifolds”, *Am. J. Math.* **85**, 79-94. (1963)
- [6] Penrose R, “Nonlinear gravitons and curved twistor theory”, *Gen. Rel. Grav.* **7**, 31–52 (1976)
- [7] Merkulov S, “Geometry of Kodaira moduli spaces”, *Proc. Amer. Math. Soc.* **124**, 1499-1506. (1996)
- [8] Merkulov S and Schwachhofer L, “Classification of irreducible holonomies of torsion-free affine connections”, *Ann. of Math.* **150**, 77-149. (1999)
- [9] Cartan E, “Sur les variétés à connexion affine et la théorie de la relativité généralisée”, *Ann. Sci. Ecole Norm. Sup. (4)* **40** (1923), 325.
- [10] Künzle H, “Covariant Newtonian limit of Lorentz space-times”, *Gen. Rel. Grav.* **7**, 445. (1976)
- [11] Dautcourt G, “On the Newtonian Limit of General Relativity”, *Acta. Phys. Pol B21*, 755. (1989)
- [12] Son D, “Newton-Cartan geometry and the quantum Hall effect”, *arXiv:1306.0638*.
- [13] Gibbons G and Hawking S, “Gravitational multi-instantons”, *Phys. Lett. B78* 430 (1978)
- [14] Salamon S, “Differential geometry of quaternionic manifolds”, *Ann. Sci. Ecole Norm. Sup.* **19** 31-55. (1986)
- [15] Pedersen H and Poon Y S, “Twistorial construction of quaternionic manifolds”, *Proceedings of the Sixth International Colloquium on Differential Geometry, Santiago de Compostela*, 207-218. (1988)
- [16] Grothendieck A, “Sur la classification des fibrés holomorphes sur la sphère de Riemann”, *Am. J. Math.* **79** 121-138 (1957)
- [17] Dunajski M, Gundry J, and Tod P, “Jumps, folds, and singularities of Kodaira moduli spaces”, *arXiv:1607.05307* (2016)
- [18] Bergshoeff E, Hartong J, and Rosseel J, “Torsional Newton–Cartan Geometry and the Schrödinger Algebra”, *Class. Quantum Grav.* **32** 13 (2015)
- [19] Gundry J, “Higher symmetries of the Schrödinger operator in Newton–Cartan geometry”, *J. Geom. Phys.* **113** (2017)
- [20] Dunajski M, “Solitons, Instantons & Twistors”, *Oxford Graduate Texts in Mathematics*, OUP (2009)

- [21] Dunajski M and Penrose R, "On the quadratic invariant of binary sextics", *Math. Proc. Cambridge Philos. Soc.* (2016)
- [22] Dunajski M and Tod P, "Paraconformal geometry of nth order ODEs, and exotic holonomy in dimension four", *J. Geom. Phys.* 56 (2006)
- [23] Bailey T and Eastwood M, "Complex Paraconformal Manifolds – their Differential Geometry and Twistor Theory", *Forum Mathematicum* 3 3 (1991)
- [24] Bryant R, "Two exotic holonomies in dimension four, path geometries, and twistor theory", *Proc. Symp. Pure. Maths. Vol. 53*, 33–88. (1991)
- [25] Dunajski M and Mason L, "Twistor theory of hyper-Kähler metrics with hidden symmetries", *J. Math. Phys.* 44 3430-3454 (2003)
- [26] Gindikin S, "Reduction of manifolds of rational curves and related problems of the theory of differential equations", *Functional Analysis and Its Applications* 18.4 (1984)
- [27] Bekaert X and Morand K, "Connections and dynamical trajectories in generalised Newton-Cartan gravity I. An intrinsic view", *J. Math. Phys.* 57 (2016)
- [28] Festuccia G, Hansen D, Hartong J, and Obers N, "Torsional Newton-Cartan geometry from the Noether procedure", *Phys. Rev. D* 94 (2016)